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KENT'S FIRST COURSE IN ALGEBRA

Refer to pages

REASON FOR THE BOOK

The solution of problems difficult or impossible
to solve by arithmetic is one of the chief aims of
algebra. Hence Professor Kent has chosen as 43-66
the central idea for his book the earliest possible 97-129
introduction of the equation and its almost immediate application to the solution of such problems.

New terms and principles are skillfully introduced as soon as they are needed to further the development of the equation, but no terms are defined or introduced until the pupil is ready to use them.

ARRANGEMENT OF THE SUBJECT MATTER

The order of topics is such as to give the pupil
at the earliest possible moment preparation for 4-6
working practical problems of genuine interest 9-10
because of their bearing on everyday life. Thus 29-32
little time is wasted on work which has even the etc.
doubtful value formerly accorded to the type of problem known as "mental gymnastics."

CORRELATION WITH OTHER BRANCHES

The vital connection between arithmetic, algebra and geometry is clearly demonstrated.

CORRELATION WITH OTHER BRANCHES—*Continued.*	Refer to pages
The processes of algebra are deduced from arithmetic by steps plainly indicated and easily followed and the facts of geometry are the basis for many of the problems of algebra.	1, 13 15, 16 53, 225 etc.
AMPLE WRITTEN AND ORAL DRILL EXERCISES	
The exercises in purely manipulative work, given only where necessary to explain and fix a principle, are confined to the simplest instances, only sufficiently complex to show the real nature of the process.	33, 99 121, etc.
Oral exercises are given when their presence is of distinct value in developing the problem at hand, or when desirable for review.	19, 98 242, etc.
KIND OF PROBLEMS USED	
Antiquated and stereotyped problems of another class have been eliminated, and bright, timely and interesting problems substituted.	
The problems given here stimulate the pupil to active interest by impressing him as being well worth while. They deal with practical subjects seen in school or at home, or with trades or professions likely to be followed by the average boy or girl upon leaving school.	65, 126 162, 219 239, etc.
REVIEW EXERCISES	
At the close of each chapter there are ample review exercises, calculated to fix firmly in the mind of the student the principles mastered in the body of the chapter.	24-28 64-66 203-206 etc.

THE NOTATION USED

Refer to pages

The symbols for notation familiarize the student with the application of algebra to other subjects. A varying system of notation is used for the purpose of stimulating the pupil to recognize and solve as readily an equation in another lettering as in x and y. 111, 132 210, etc.

SIMPLIFICATION OF FACTORING

Factoring is simplified by the use of short, concise explanations, admirably illustrated by brief but well selected written exercises. 103, 109 119, etc.

EARLY INTRODUCTION OF QUADRATICS

Quadratic equations are introduced in the sixth chapter, rather earlier than in the older texts, and factoring has been applied to their solution. 97-129

CHECKING RESULTS

Insistence is made upon the checking of results. Persistent attention to this matter will reduce greatly the percentage of inaccuracy. Methods of checking are explained and illustrated. 34, 48 72, 158 etc.

AS PREPARATION FOR COLLEGE

For the student planning to attend college the book provides ample preparation. In addition to college requirements the recommendations of the different committees and associations of mathematics teachers have been carefully met.

FOR PUPILS NOT GOING TO COLLEGE

Those who are unable to attend college will find Kent's Algebra excellent preparation for a business career, since the problems and exercises touch upon and open up so many avenues of thought bearing on the practical business life which such pupils are soon to enter.

Refer to pages

22, 50
161, 205
217, etc.

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A FIRST COURSE IN
ALGEBRA

A FIRST COURSE IN ALGEBRA

BY

FREDERICK C. KENT

ASSOCIATE PROFESSOR OF MATHEMATICS, UNIVERSITY OF OKLAHOMA

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PREFACE

Two ends have been kept in view in the preparation of *A First Course in Algebra.* First, to give the student who will take no more algebra a good general working knowledge of the subject sufficient for any elementary course in science, industrial life, business, etc.; second, to furnish a thorough foundation for the more advanced courses of those who are to continue their studies in the university.

That one of the chief aims of algebra is the solution of problems which are either difficult or impossible to solve by arithmetic, and that the equation is the means used in obtaining such solutions, are prominent features of this text. In fact, the equation is made the central idea of the course. The order of topics is such as to employ a principle as soon as it is mastered in the solution of equations. Care has been exercised, however, not to break entirely with conventional usage nor to carry the central idea to an extreme by sacrificing too much in order to reach some new type of equation.

To subordinate mere manipulation of algebraic symbols and processes to the more important task of solving equations and working problems; to produce a text that will be concise and yet require much activity on the part of the student; to give the application of each principle to the solution of problems immediately after its explanation;

and, finally, to stimulate the student's interest in the subject of algebra by stressing its utility when applied to the solution of the practical problems of science and everyday life,—have been some of the aims and hopes of the author.

F. C. K.

SUGGESTIONS TO TEACHERS

---

(1) Insist upon the thorough memorizing of all black-faced formulas, designated by Roman numerals. These are numbered consecutively from I to XXXI, and express in a concise and intelligible form the fundamental theorems of the subject (e.g., VIII, p. 97).

(2) All expressions in italics should be studied thoroughly but not memorized.

(3) Insist upon the checking of results. It often happens that a student will learn fully as much by the checking of a result as by the working of the problem itself. Persistent attention to this matter will reduce materially the inaccuracy so often manifest in the work of high school students. Methods of checking are explained and illustrated in connection with each operation.

(4) At least one written problem should be given every day. In working problems before the class, students should be required to write and explain at the same time.

(5) Require students to keep the solution of every problem in a note book. (Problem, as here used, means a statement capable of being expressed as an equation whose solution is desired.)

(6) Wherever it is desired to have a simple treatment of algebraic fractions early in the course, Chapter IX, which deals with fractions, may be taken up before Chapter VII.

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A FIRST COURSE IN ALGEBRA

CHAPTER I

INTRODUCTION

1. Arithmetic deals with numbers, and it is here assumed that the student is already acquainted with two kinds of numbers: integers, as 1, 2, 3, 4, and fractions, as $\frac{1}{2}$, $\frac{2}{3}$, .5, etc. In algebra the student will not only increase his knowledge of these numbers, but will also become acquainted with other kinds of numbers unknown to arithmetic. Thus algebra may be considered as a continuation of arithmetic.

2. Number Symbols. Numbers in algebra are represented by the Hindu symbols, 1, 2, 3, etc., that are used in arithmetic; also by letters, as a, b, c, x, y, z.

3. Symbols of Operation are the same in algebra as in arithmetic. Thus, the signs $+$, $-$, $\times$, and $\div$ denote addition, subtraction, multiplication, and division, respectively.

Besides the foregoing, algebra has certain symbols of operation of its own. The dot $\cdot$ when written between two numbers and above the line, signifies their product. Thus $a \cdot b$ means means $a \times b$ and is read a times b.

A product is also indicated by writing together consecutively letters, or letters and numerals with no sign between them. For example, 3 ab means 3 times a times b, abc means the product of a, b, and c, or $a \times b \times c$.

4. Summary. We have learned that (1) letters stand for numbers; (2) a dot · is a symbol for multiplication, the same as the ×; (3) the absence of a symbol between two letters means that the letters are to be multiplied. Thus $a \times b = a \cdot b = ab$.

EXERCISE

5. If x, y, and z are three numbers, express in words the meaning of the following:

1. $2x$; $4y$; $5z$.
2. $x + y + z$; $x + y - z$; $x - y - z$.
3. $4 \cdot x \cdot y \cdot z$; $4xy - 5yz$.

Suppose $x = 5$, $y = 4$, and $z = 1$ in the following:

4. What number does $x + y + z$ represent?
5. What number does $x - y + z$ represent?
6. What number does $2x + 4y - 5z$ represent?

In the following assume that x, y, and z have the same values as stated above and that $a = 10$, $b = 2$, and $c = 3$. What number is represented by each of the expressions:

7. $a + b$?

8. $\dfrac{a+b}{a-b}$?

9. $4xyz + 5abc$?

10. $\dfrac{4xy}{x-y}$?

11. $\dfrac{2a+b}{x-c}$?

12. $\dfrac{5x-y}{a+b}$?

13. $\dfrac{4ab + 2ac - 3bc}{2xy \cdot 3ab}$?

14. $\dfrac{4xy}{x-y} + \dfrac{2a+b}{2x+z} - \dfrac{2x-y}{2c}$?

15. If i stands for interest, p for principal, r for rate, and t for time, express in words the meaning of the following:

$$i = \frac{prt}{100}.$$

16. With the same meaning for the letters as in Example 15, what is the meaning of the following expression:

$$r = \frac{100\,i}{pt}?$$

17. In Example 15, suppose $p = 1000$, $r = 4$, and $t = 3$, what is the value of i?

18. In Example 16, suppose $i = 50$, $p = 500$, and $t = 2$, what is the value of r?

19. The sum of two numbers is s and one of the numbers is 5. What is the other number?

20. One number is five times another. Using n to represent the smaller number, what expression represents their sum? Their difference?

21. A man walked for 6 hours at the rate of x miles an hour. How far did he walk?

22. A boy ran x rods in 10 seconds. How many rods did he run in one second?

23. The length of a rectangle is twice its width. Representing its width as x feet, what is the perimeter (distance around) of the rectangle?

24. The width of a rectangular lot is $\frac{1}{3}$ its length. Represent its length by $3\,x$ rods; then what will be the distance around the lot?

25. A horse cost three times as much as a cow and the cow cost twice as much as a sheep, which cost c dollars. What did the horse cost?

26. The first of three consecutive numbers * is n. What are the other two numbers?

---

* Consecutive numbers are such as 3, 4, 5, or 10, 11, 12.

27. Out of an annual income of $2000 a man saved x dollars the first year, twice that amount the second year, and four times as much the third year as for the second. How much did he save the third year?

28. A railway train after running x miles an hour for three hours increased its speed by 5 miles an hour for the remaining four hours of its trip. What was the length of the whole trip?

29. The circumference of a circle is $2\pi x$ feet. What is its radius? (Circumference $= 2\pi$ times radius.)

30. The radius of a circle is y feet. What is the circumference?

6. One of the Aims of Algebra is the solution of problems. Many problems whose solution is impossible or very difficult by arithmetic are easily solved by algebra.

7. Axioms. Every step taken in the solution of a problem depends on one or more elementary statements agreed to by universal consent and called *axioms* or *assumptions*. These are:

1. If equals be added to equals, the sums will be equal.
2. If equals be taken from equals the differences will be equal.
3. If equals be multiplied by equals the products will be equal.
4. If equals be divided by equals, the quotients will be equal.
5. Things equal to the same thing are equal to each other.

8. Illustrative Solutions of Problems.

Problem 1. The sum of two numbers is 40, and the greater is four times the smaller. What are the numbers?

Observe that two conditions are stated in this problem. First, that the sum of the two numbers is 40; second, that the greater number is four times the smaller.

Solution in Words

One of the numbers is the smaller number.

Four times that number equals the greater number. (2d condition.)

Five times that number equals the sum of both numbers.

Forty equals the sum of both numbers. (1st condition.)

Therefore, five times that number equals forty. (Axiom 5: Things equal to same thing . . .)

Then, the smaller number equals 8, $\frac{1}{5}$ of 40;

and the greater number equals thirty-two, four times eight.

Abbreviated or Algebraic Solution

Let s = the smaller number;

then $4s$ = the greater number, (2d condition)

and $5s$ = the sum of both numbers, since $s + 4s = 5s$.

But, 40 = the sum of both numbers. (1st condition.)

Therefore, $5s = 40$. (Axiom 5.)

Then, $s = 8$, dividing both sides of last equation by 5; (Axiom 4: Equals divided by equals . . . .)

and $4s = 32$, multiplying both sides of preceding equation by 4. (Axiom 3: Equals multiplied by equals . . .)

Therefore the smaller number is 8 and the greater number is 32.

Check. $32 + 8 = 40$, 1st condition;

and $32 = 4 \times 8$, 2d condition.

It is very important to test answers to see if they satisfy every condition stated in the problem.

Problem 2. The perimeter of a rectangular lot is 240 ft. and the length of the lot is three times its width. Find its dimensions.

FIG. 1

The conditions stated in this problem are:

1st, The perimeter equals 240 ft.

2d, The length equals three times the width.

Algebraic Solution

Let w = number of ft. in the width;

then $3w$ = number of ft. in the length, (2d condition)

and $8w$ = number of ft. in the perimeter, since

$$w + 3w + w + 3w = 8w.$$

But 240 = number of ft. in the perimeter. (1st condition.)

Therefore, $8w = 240$. (Axiom 5.)

Then $w = 30$, dividing both sides of the preceding equation by 8, (Axiom 4)

and $3w = 90$, multiplying both sides of the preceding equation by 3. (Axiom 3.)

Hence the dimensions of the lot are, width 30 ft., length 90 ft.

Check. $30 + 90 + 30 + 90 = 240$, 1st condition;

and $90 = 3 \times 30$, 2d condition.

Problem 3. What sum of money (principal) placed at interest for two years at 5% will amount to $220, simple interest?

Let p represent the principal and i the interest,

then $i = .10\,p\left(\text{using the formula } i = \dfrac{prt}{100}\right)\cdot$ (1st condition).

and $p + .10\,p = 220$ (since from the 2d condition $i + p = 220$).
Collecting, $1.10\,p = 220$.
Therefore, $p = 200$ (dividing both sides by 1.10).

The required sum, therefore, is $200.

PROBLEMS

9. Solve each of the following problems in a manner similar to the foregoing and check results:

1. A man sold a horse and buggy for $300, receiving twice as much for the horse as for the buggy. How much did he receive for each?

2. The greater of two numbers is 9 times the less, and their sum is 70. What are the numbers?

3. A number increased by three times itself, four times itself, and five times itself, becomes 65. What is the number?

4. A, B, and C engage in business with a combined capital of $30,000. If A furnishes twice as much capital as B, and C three times as much as B, how much capital does each furnish?

5. There are 72 men, women and children in a room. The number of men is twice the number of women, and the number of children is twice the number of men and women together. How many children are there in the room?

6. A line 72 inches in length is divided into three parts so that the longest part is three times the length of the shortest and the second part is twice the length of the shortest. Find the length of each part.

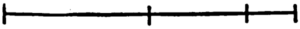

FIG. 2

7. Divide $80 among three boys so that the first shall have three times as much as each of the other two.

8. A school containing 630 pupils was classified according to ages. It was found that there were three times as many pupils between the ages of 12 and 16 as there were between the ages of 16 and 18; and twice as many between the ages of 16 and 18 as between the ages of 18 and 20. How many pupils belonged to each group?

9. It is proved in geometry that the sum of the three angles of a triangle is 180°. In triangle *ABC*, angle *C* is three times angle *A*; and angle *B* is twice as large as angle *A*. How many degrees are there in each angle?

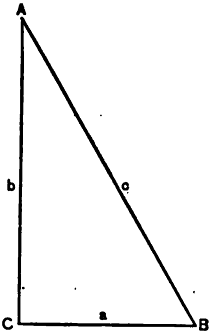

FIG. 3

10. A farmer raised 3000 bushels of wheat, corn, and oats. The number of bushels of wheat was three times the number of bushels of corn, and the number of bushels of oats was four times the number of bushels of corn. How many bushels of each kind of grain did he produce?

11. The population of the United States, in 1912, is in round numbers 105,000,000. Grouped according to chief occupations, it is found that the number belonging to the largest group, agriculture, is four times the number belonging to the mining group; twice the number belonging to the manufacturing group; and six times the number belonging to all the other groups combined. What is the population of the agricultural group?

12. The population of New York City is five times that of St. Louis and twice that of Chicago. The combined population of all three cities is 7,650,000. Find the population of each city. (Suggestion: Let $2x$ = population of St. Louis; then $10x$ = ?).

13. A rectangle is four times as long as it is wide and the distance around it is 200 rods. Find its dimensions.

14. The sum of the three sides of triangle *ABC* is 40 inches. If side *BC* is twice as long as side *AC*, and side *AB* is also twice as long as the side *AC*, how long is each side?

FIG. 4

15. A man has 240 feet of wire fence with which he wishes to enclose a piece of ground, rectangular in shape with length twice the width. What will be the width of the largest piece of ground that he can enclose with his fence under these conditions?

16. What sum of money placed at interest, at 7% simple interest, will amount to $363 in three years?

17. The regulation foot-ball field is 140 feet longer than it is wide, and the sum of its length and width is 460 feet. What are its dimensions?

18. A newsboy sold a certain number of papers on Monday and on each succeeding day of the week increased his sales twofold. If the total sales from Monday to Saturday were 6300 papers, how many did he sell on Saturday?

19. The perimeter of a rectangle formed by placing three equal squares in a row is 64 inches. What is the side of one of these squares?

20. A and B are 100 miles apart and travel toward each other until they meet, when it is found that A's rate of travelling has been twice that of B; also that A has been twice as long on the trip as B, having started first. How many of the 100 miles did each cover?

10. A Term in Algebra is a single group of number symbols whose parts are not separated by a sign + or −; as, 2 *abc*, which means the product of 2 times *a* times *b* times *c*.

11. Factors. If two or more numbers are multiplied together, each of the numbers is called a *factor*; the result is called a *product*. Thus, in 2 *abc*, each of the numbers 2, *a*, *b*, and *c* is a factor, and 2 *abc* is a product. It should be observed that 2 *abc* is also a number.

12. Coefficient. Any factor of a term is a *coefficient* of that term. Coefficients may be numerical or literal. Thus, in the term 2 *abc*, 2 is the numerical coefficient of *abc*; 2 *a* is the coefficient of *bc*, and 2 *ab* is the coefficient of *c*.

13. A Power is a product obtained by multiplying a factor by itself one or more times. A factor thus used is called a *base*. Thus, 8 is the third power of base 2; 9 is the second power of base 3.

14. An Exponent, when integral, is a number which shows how many times a base is to be multiplied by itself. It is written just above and to the right of the base. Thus in the term y^3, y is the base and the exponent is 3. It shows that y is to be multiplied by itself three times and means the same as $y \cdot y \cdot y$ or y times y times y. Like the coefficient, the exponent may be either numerical or literal. In the term $2\,a^x$, x is the exponent of a, and means that a is to be used as a factor x times, and that the product thus obtained is then to be multiplied by 2. When no exponent is expressed an exponent (1) is understood; as in $3\,x$, which means the same as $3\,x^1$.

15. An Algebraic Expression is a group of terms connected by + and − signs. An expression consisting of two or more terms is called a *polynomial*. But a polynomial consisting of two terms is often called a *binomial*, and a polynomial of three terms is also called a *trinomial*. Thus $x + y$ is a binomial; $x + y + z$, a trinomial; while both the foregoing and expressions consisting of more than three terms are polynomials.

EXERCISE

16. Find the value of:

1. $3\,x^3$, if $x = 2$; if $x = 3$; if $x = 5$.
2. $5\,x^3y^2$, if $x = 3$ and $y = 2$; if $x = 2$ and $y = 3$.
3. $x^3 + y^2$, if $x = 3$ and $y = 2$.
4. $\frac{6}{x} - \frac{6}{y}$, if $x = 2$ and $y = 3$.

5. $\dfrac{a^3 + b^3}{a + b}$, if $a = 3$ and $b = 2$.

6. Show that $\dfrac{a^3 + b^3}{a + b} = a^2 - ab + b^2$ is true if $a = 3$ and $b = 2$.

7. Show that $\dfrac{a^3 + b^3}{a + b} = \dfrac{a^3}{a + b} + \dfrac{b^3}{a + b}$ is true if $a = 3$ and $b = 2$.

8. * Show that $\dfrac{1}{a} + \dfrac{1}{b} \neq \dfrac{1}{a + b}$ when $a = 3$ and $b = 2$.

Try other values.

9. $x = \dfrac{ab}{2}$. Find the value of x, if $a = 5$ and $b = 4$.

10. $s = \dfrac{1}{2} gt^2$. Find the value of s, if $g = 32$ and $t = 5$.

11. $s = 2\pi RL + 2\pi R^2$. Find the value of s, if $\pi = 3.1416$, $R = 10$, and $L = 12$.

12. $V = \pi R^2 L$. Find the value of V using the same values for π, R, and L as in Example 11.

13. $\dfrac{a^x + b^y}{a + b}$, if $a = 3$, $b = 4$, $x = 5$, and $y = 2$.

14. $\dfrac{6\,a^x b^y \cdot a^y y}{2\,x^a y^b}$, using the same values of letters as in Example 13.

15. A body falling freely from a state of rest will pass through a space $s = \dfrac{1}{2} gt^2$ in t seconds. The value of s then in Example 10 above is the space through which such a falling body passes in 5 seconds. Assuming the same value of g (32 ft.) find the space passed over by a body falling for three seconds. ($t = 3$.)

16. $V = \pi R^2 L$ is the formula for finding the volume (V) of a cylinder. The value of V in Example 12 is the volume of a cylinder whose length is 12 and whose base is a circle with radius 10. Assuming the same value for π as in Example 11, find

---

* Sign $\neq$ means "is not equal to."

the volume of a cylinder whose length is 20 feet and the diameter of whose base is 8 feet.

17. What does $x^2 - 5x + 6$ equal (*a*) if $x = 2$? (*b*) if $x = 3$? (*c*) if $x = 4$?

18. What does $3x^2 - 14x - 5$ equal, (*a*) if $x = 5$? (*b*) if $x = 6$?

Write out in words the meaning of the expressions in Examples 19 and 20.

19. (*a*) $5b^3$; (*b*) $\frac{x^2 + y^2}{3}$; (*c*) 3^a.

20. (*a*) $3b^4c^3$; (*b*) $\frac{1}{4}b^2c^4$; (*c*) 5^e.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

17. Double Meaning of Numbers. In many practical affairs we must deal with numbers that have directly opposite meanings.

For example, on the scale of a thermometer it will be observed that the temperature is read both ways, or in opposite directions from zero. The scale is read upward for temperatures above zero and downward for temperatures below zero. Hence the numbers on one side of zero of this scale are directly opposed in meaning to the numbers on the other side of zero.

Similarly, in geography, latitude is reckoned north and south, that is, in opposite directions, from the earth's equator. Longitude is reckoned east and west from a given meridian. Hence, 30° north latitude is opposite in meaning to 30° south latitude; 10° west longitude to 10° east longitude.

Again, in business transactions a loss is regarded as the opposite to a gain; a liability as the opposite to an asset.

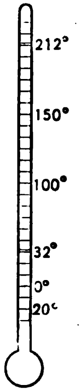

FIG. 5

It therefore becomes necessary for us to enlarge upon our conception of the numbers of arithmetic which supposes all numbers to be of the same meaning or sense and include numbers of opposite meaning.

18. Positive and Negative are terms used to denote numbers of opposite meaning. It is immaterial which number is to be considered as positive or which negative. But it is very essential to note that, having designated a certain number as positive, a number having the opposite meaning must be considered as negative.

Thus, if we designate the numbers above zero on the thermometer scale as positive, the numbers below zero must be regarded as negative. And, if we regard north latitude as being positive, then south latitude must be regarded as negative.

19. Signs + and −, as already explained, are used in algebra and in arithmetic as symbols of operation to indicate addition and subtraction respectively. These signs are also used in algebra to indicate whether a number is to be considered as positive or negative. If a number is *negative*, that fact is *always* indicated by placing a − sign before it. The sign + in like manner shows that the number before which it stands is positive. In practice, however, the + sign is omitted and all numbers not specifically marked with a − sign are to be considered as positive.

ORAL EXERCISE

20. 1. Regarding temperatures above zero as positive, what is meant by a temperature of − 80°?

2. Considering north latitude as positive, what is meant by a latitude of − 20°?

3. Dates after the birth of Christ are taken as being positive. What does the date − 776 mean?

4. Two ships are in the same longitude and one is in latitude + 10°, the other in latitude − 15°. In what direction is the second from the first?

5. Longitude is usually reckoned east and west from the meridian of Greenwich. Considering east longitude as positive, in which direction from Greenwich, east or west, does a place lie whose longitude is − 12°?

6. If gains be regarded as positive, what does a gain of − $5 mean? A loss of − $5?

7. A man's property is worth \$2000 and his debts amount to \$1000. How may positive and negative numbers be used to represent each of these amounts?

8. During the first five minutes of a football game the team in possession of the ball advanced it as follows:
$+30$ yds., $+5$ yds., $+10$ yds., -4 yds., -3 yds., $+2$ yds.
Explain the movements of the ball.

9. During a certain interval of playing a football was advanced 20 yds., then 10 yds. more, then carried back 5 yds. by reason of a penalty. How may positive and negative numbers be used to represent these movements of the ball?

10. Rome was founded in 733 B.C. and the Roman Empire fell in 476 A.D. How may positive and negative numbers be used to express these dates?

21. Number Scale of Arithmetic. Let us suppose that some convenient unit of length, as $\frac{1}{4}$ of an inch, is laid off successively along a given straight line and that the successive points of division are designated with the natural numbers as follows:

I.

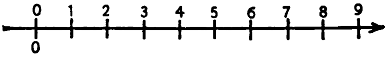

FIG. 6

Beginning at zero and proceeding along the line towards the right, it will be observed that: (1) each number in the series expresses the distance (in terms of the assumed unit of length) of the point it designates from the point designated by zero. Thus, the point designated by 3 lies 3 units to the right of zero; (2) the series represents an orderly succession of increasing values as we proceed towards the right and an orderly succession of decreasing values as we proceed towards the left; (3) this series may be extended indefinitely towards the right, that is, in the direction of increasing values.

This series of numbers, designated I, constitutes the *number scale of arithmetic.*

22. Number Scale of Algebra. Let us now suppose that (1) we have a given straight line whose middle point is designated 0. (2) Some convenient unit of length as before is laid off successively along the given line in each of the two opposite directions from zero. (3) Distances to the right of zero are to be considered positive. (4) The successive points of division are designated as follows:

II.

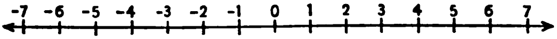

FIG. 7

Beginning at zero and proceeding along the line in either of the two opposite directions, it will be observed that (1) each number in the series expresses the distance as before of the point it designates from zero. Thus, the point designated by 3 lies 3 units to the right of zero and the point designated by -3 lies 3 units to the left of zero. (2) Since distances to the right of zero are positive, distances to the left of zero, being measured in the opposite direction, must be regarded as negative; and the numbers which express these distances to the left of zero are therefore negative numbers and must be preceded by the $-$ sign. (3) This series represents an orderly succession of increasing values as we proceed towards the right and an orderly succession of decreasing values as we proceed towards the left. It is because of this fact that negative numbers are sometimes spoken of as being less than zero. (4) This series may be extended indefinitely both to the right and to the left of zero and therefore has double the scope of series I.

The series designated II constitutes the *number scale of algebra.*

EXERCISE

23. 1. On a horizontal straight line as $X'OX$, consider the part OX as being positive; the part OX' as negative; and locate points corresponding to the following numbers: -3, $+7$, $-2\frac{1}{2}$, $+3\frac{1}{4}$, $-6\frac{1}{4}$.

2. Locate the points corresponding to the numbers in Example 1 on a vertical line as YOY', considering OY as the positive and OY' as the negative parts of the line.

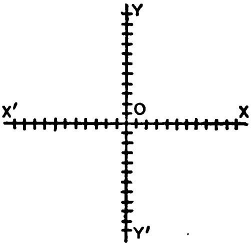

FIG. 8

3. Let P be a point which, starting at O, moves along the line $X'OX$, of Example 1, as follows:

(*a*) 10 units in the positive direction, then 2 units in the negative direction.

(*b*) 10 units in the negative direction, then 15 units in the positive direction.

Trace the movements of P in each instance and designate its positions on the line $X'OX$ by some letter as P',* P'', etc.

4. Let the point P start at O and move along the vertical line YOY' of Example 2 as follows:

(*a*) 6 units in the positive direction and then 12 units in the negative direction.

(*b*) 4 units in the positive direction, then 5 more units in the positive direction, and then 3 units in the negative direction. Trace the movements of P in each of these instances and designate its position by P''' P^{iv}, P^{v}.

5. Referring to the movements of P in Example 3, how many units of distance does P pass over in (*a*)? In (*b*)?

6. Referring to the movements of P in Example 4, over how many units of distance does P pass in (*a*)? In (*b*)?

7. Referring to the positions of P in Example 3, where does P lie with reference to O at the end of the movements in (*a*)? In (*b*)?

---

* P' is read P prime; P'' is read P second, etc.

8. Referring to the positions of *P* in Example 4, where does *P* lie with respect to *O* at the end of the movements in (*a*)? In (*b*)?

9. What net result was obtained by combining + 10 units distance and − 2 units distance in Example 3*a*? By combining − 10 units distance and + 15 units distance in Example 3*b*?

10. What net result was obtained by combining + 6 units distance and − 12 units distance in Example 4*a*? By combining + 4 units distance with + 5 units distance and − 3 units distance in Example 4*b*?

24. Addition of Positive and Negative Numbers may be readily performed by the use of the number scale of algebra.

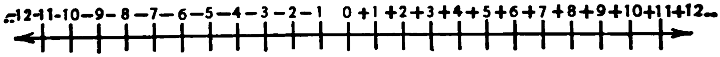

FIG. 9

To add + 4 and + 8, begin at + 8 and go 4 units in the direction of positives, to the right. This will bring you to + 12, which is the sum of + 4 and + 8.

To add + 4 to − 8, begin at − 8 and go 4 units in the direction of positives. This will bring you to − 4, which is the sum of + 4 and − 8.

In general, to add the positive number *a* to the number *b*, when *b* is either positive or negative, begin at *b* and go *a* units in the direction of positives, to the right.

Since, to add + 4 to + 8 means that you begin at + 8 and go 4 units in the direction of positives, to add − 4 to + 8, begin at + 8 as before but go 4 units in the direction of negatives, to the left. This will bring you to + 4, which is the sum of − 4 and + 8.

To add − 4 to − 8, begin at − 8 and go 4 units in the direction of negatives, to the left. This will bring you to − 12, which is the sum of − 4 and − 8.

In general, to add a negative number a to a number b, when b is either positive or negative, begin at b and go a units in the direction of negatives, to the left.

25. Summary. The results obtained in the preceding section may be summarized as follows:

(1) *To add numbers which have the same direction, add as in arithmetic. If the numbers to be added are positive, the sum will be positive; if the numbers to be added are negative, the sum will be negative.*

(2) *To add two numbers which have different directions, subtract the numbers. If the greater number is positive, the sum will be positive; if the greater number is negative, the sum will be negative.*

(3) *To add more than two numbers which have different directions, first, add those which have the same direction as in paragraph* (1); *second, add these two results as in paragraph* (2).

ORAL EXERCISE *

26. 1. Add 4 to 2.

2. Add 4 to -2.

3. Add -4 to $+2$.

4. Add -4 to -2.

5. If the temperature is now $-10°$, what will it be after a rise (*a*) of 5°? (*b*) of 15°?

6. A ship is in latitude $-20°$ and is sailing northward at the rate of 5° of latitude per day. What will its latitude be (*a*) at the end of 3 days? (*b*) at the end of 5 days?

7. A ship is now in longitude 77° †. What will be its longitude after it has sailed (*a*) eastward 50°? (*b*) westward 50°?

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* To the Teacher: Pupils find that a model made from a yardstick with a rider to slide back and forth is very helpful at this point in the addition and subtraction of positive and negative numbers.

† Consider east longitude as positive.

8. If the temperature is now 15°, what will it be after (*a*) a rise of 20°? (*b*) a fall of 20°?

9. The poet Horace was born in the year -65 and died at the age of 57. In what year did he die?

10. Suppose that a man is worth $-\$1000$ and that he saves \$500 a year for 4 years. What will he then be worth?

27. Subtraction of Positive and Negative Numbers may be performed readily by the use of the number scale of algebra.

Illustrations: Since to add $+4$ to $+8$ means that you begin at $+8$ on the scale and go 4 units to the right, to subtract $+4$ from $+8$, begin at $+8$ and go 4 units to the left. This will bring you to $+4$, which is the difference between $+8$ and $+4$.

Since to add $+4$ to -8, means that you begin at -8 and go 4 units to the right, to subtract $+4$ from -8, begin at -8 and go 4 units to the left. This will bring you to -12, which is the difference between -8 and $+4$.

Since to add -4 to -8 means that you begin at -8 and go 4 units to the left, to subtract -4 from -8 begin at -8 and go 4 units to the right. This will bring you to -4, which is the difference between -8 and -4.

To subtract a positive number a from a number b, b being either positive or negative, begin at b and go a units to the left; to subtract a negative number a from a number b, b being either positive or negative, begin at b and go a units to the right.

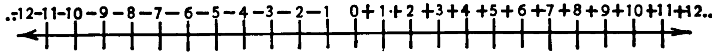

FIG. 10

From the foregoing it will be observed that (1) to subtract a positive number is the same as to add a negative; (2) to subtract a negative gives the same result as to add a positive. Hence, the following is a convenient rule for the subtraction of positive and negative numbers:

Change the sign of the number to be subtracted and add to the other number.

EXERCISE

1. Subtract (*a*) 2 from 5. (*b*) 5 from 2.

2. Subtract (*a*) 3 from 7. (*b*) 7 from 3.

3. If the temperature is now + 30°, what will it be after a fall (*a*) of 20°? (*b*) 40°?

4. If the temperature is now + 10°, what will it be after a fall of (*a*) 12°? (*b*) 8°?

5. A ship is in latitude + 40° *. What will be its latitude after it sails southward (*a*) 25°? (*b*) 50°?

6. A ship is in longitude + 77°. What will be its longitude after it sails westward (*a*) 50°? (*b*) 92°?

7. A man's gains in a certain business amounted to $3000 and his losses for the same period amounted to $2000. What was his net gain?

8. Supposing the gains and losses in Example 7 to be reversed, what would have been his net gain?

9. A man's annual income is $2000 and his expenses amount to $1000 a year. What are his annual savings?

10. Suppose the income and expenses of Example 9 to be reversed. What would his annual savings be?

11. From the sum of $-3\frac{1}{2}$, $-18\frac{1}{4}$, and $33\frac{3}{4}$ subtract the sum of 18, $9\frac{1}{4}$, and $33\frac{1}{4}$.†

---

* Hereafter it will be assumed that (*a*) north latitude is positive and south latitude negative; (*b*) east longitude is positive, west longitude negative.

† Add the first and second numbers first, then add their sum to the third number.

12. A steamer's course begins at Boston in latitude 42° N. and ends at Buenos Ayres in latitude 32° S. How many degrees of latitude does the course cover?

13. $-32°$ subtracted from 42° leaves how many degrees? 42° subtracted from $-32°$ leaves how many degrees?

14. At 12 noon, December 1, the temperature at a certain city was 20° above zero. At 7 A.M., December 2, the temperature had fallen 41° lower than what it was at noon of the preceding day. What was the temperature at 7 A.M. December 2? $20° - 41° = ?$ degrees?

15. Suppose the temperature at noon of December 2, to be 45°. How much of a rise would be necessary above the temperature indicated at 7 A.M. in the preceding example?

16. In levelling a piece of ground a surveyor selected a certain point as base and with reference to this base found the height of six other points to be $+35$ feet, -16 feet, -9 feet, $+24$ feet, -18 feet, $+4$ feet. What was the average height of the six points?

17. A launch has a speed of 10 miles an hour in still water. What will be its rate both up and down a river whose current flows 2 miles per hour? (Use positive and negative numbers to express the rate of the current.)

18. On the deck of a boat whose speed is 20 miles an hour two men are walking, each at the rate of 4 miles an hour, one going in the same direction as the boat and the other in the opposite direction. Use positive and negative numbers to represent the rates of walking and express the rate at which each of these men is advancing towards his destination.

19. A boy riding along the street on a motorcycle at the rate of 40 miles an hour encounters a gust of wind blowing in the opposite direction at the rate of 60 miles an hour. (*a*) At what rate does the wind pass him? (*b*) At what rate would the wind pass him if both were going in the same direction?

20. The daily papers usually carry a table furnished by the weather bureau showing maximum and minimum temperatures at different points? How would the following readings be printed in such a table? St. Paul, 10° below zero, Chicago, 2° below zero, Memphis, 6° above zero, New Orleans, 30° above zero?

28. Multiplication of Positive and Negative Numbers.

Illustrations

$+3 \times +8 = +24$, since this means

$$(+8) + (+8) + (+8) = +24.$$

$+3 \times -8 = -24$, since this means

$$(-8) + (-8) + (-8) = -24.$$

$-3 \times +8 = -24$, since this means that $+8$ is to be subtracted three times. This means the same thing as subtracting 24 once, which gives -24.

$-3 \times -8 = +24$, since this means that -8 is to be subtracted three times. This means the same as subtracting -24 once which gives $+24$.

In general, two numbers a and b, both being positive or both being negative, when multiplied together give a positive result, $+ab$. And, two numbers a and b, either of which may be positive but the other negative, when multiplied together, give a negative result, $-ab$.

ORAL EXERCISE

Give results to the following:

1. $+3 \times +4$.
2. $+3 \times -4$.
3. $-5 \times +6$.
4. -5×-6.
5. $+4 \times +3 \times +2$.
6.* $+4 \times +3 \times -2$.
7. $+4 \times -2 \times +3$.
8. $+4 \times -2 \times -3$.
9. $-4 \times -2 \times +3$.
10. $-4 \times -2 \times -3$.

---

* Multiply $+3$ by $+4$, then multiply -2 by this result.

29. Division of Positive and Negative Numbers.

Illustrations

$$
\begin{aligned}
+8 \div +4 &= +2, \text{ since } +4 \times +2 = +8.\\
-8 \div +4 &= -2, \text{ since } +4 \times -2 = -8.\\
+8 \div -4 &= -2, \text{ since } -4 \times -2 = +8.\\
-8 \div -4 &= +2, \text{ since } -4 \times +2 = -8.
\end{aligned}
$$

In general, two numbers a and b, both being positive or both negative, when divided, give a positive result. And, two numbers a and b, either of which may be positive but the other negative, when divided, give a negative result.

30. Summary. The results reached in the two preceding sections lead to the following law of signs for multiplication and division:

In multiplication and division of positive and negative numbers, like signs produce positive results and unlike signs produce negative results.

ORAL EXERCISE

31. 1. $+30 \div +3 =$ 2. $-30 \div +3 =$

3. $+30 \div -3 =$ 4. $-30 \div -3 =$

5. $-60 \div +5 =$ 6. $-60 \div +12 =$

7. $-60 \div -12 =$ 8. $+60 \div -5 =$

9. $-10 \div -10 =$ 10. $-20 \div -40 =$

MISCELLANEOUS EXERCISES FOR REVIEW

32. Simplify the following expressions by performing the operations indicated:

1. $-9-(4)$. 2. $-9-4$.

3. $-11+(-13)$. 4. $-6-(-8)$.

5. $7 - 0$. **6.** $0 - 3$.

7. $0 - 2$. **8.** $8(-3)$.

9. $-3(18)$. **10.** $-5(4)$.

11. $-5(0)$. **12.** $0(-5)$.

13. $-4(7) \div -2$. **14.** $(-3)^2$.

15. $-(3)^2$. **16.** $2(3)(-8) \div (-8)$.

17. $6 + 3(2) + 18 \div (-3)$.*

18. $3(6) \div 9 - 2(6) + 4 + (-3)$.

19. $-4(-5) + (-10) \div (-3) - 4(5) \div (-2)(5)$.

20. $6(-5) + (-3)(-8) + 2(7) \div (-14)$.

Evaluate the following expressions when $a = 10$, $b = 5$, $c = 2$, and $e = -3$:

21. $\dfrac{5a^2 - 2abc}{3ac}$. **22.** $-a^2b^2c^2 + 50abc$.

23. $e^c + c^b - a^2$. **24.** $\dfrac{abc + 2ab - a^2c^2}{-2bc}$.

25. $\dfrac{1}{a} - \dfrac{1}{b} - \dfrac{1}{c}$.

26. Show that for the above values of a and b $\dfrac{a^2 - b^2}{a - b} = a + b$.

27. Show that $\dfrac{a^2 - 2ab + b^2}{a - b} = a - b$, for same values of a and b as above.

28. Show that $\dfrac{b^3 - c^3}{b - c} = b^2 + bc + c^2$, for same values of a, b, and c as above.

29. Show that $\dfrac{a^3 + c^3}{a + c} = a^2 - ac + c^2$, for same values of a and c as above.

---

* In a series of operations involving addition, subtraction, multiplication, and division, multiplication and division always precede addition and subtraction. In this example first divide 18 by -3, then perform the addition.

As stated elsewhere it is the custom to regard temperature below zero as − and temperature above zero as +; in the exercises that follow latitude north will be considered +, distance traveled southward as −; time before the birth of Christ will be considered as −, time after the birth of Christ as +; a rise in temperature will be taken as +, a fall in temperature as −. Working under these assumptions answer the following:

30. If the temperature on a given morning was − 12°, what would it be after a rise of 15°? of 10°? After a rise of 30° followed by a fall of 32°?

31. If a ship was in latitude 21° and sailed southward on its course for three days covering an average of 8° of latitude each day, in what latitude was it at the end of the third day?

32. Ahmes, an Egyptian writer, who lived about 1700 B.C., is the earliest known writer to give the solution of simple equations. The earliest writer on algebra, however, is the Egyptian, Diophantus, who lived about 300 A.D. How many years between the writings of these two men?

33. The Roman historian, Livy, died in the year 17 A.D. at the age of 66. In what year was he born?

34. Rome was founded in 733 B.C. and the Roman Empire fell in 476 A.D. How many years elapsed between these two events?

35. In what latitude is a man whose position is described as being − 5 degrees north of a certain place? How many miles north of his starting point is a man who travels 9 degrees south and then 4 degrees north? (A degree = 69.16 miles.)

36. Starting from the middle of the field in a certain football game, the ball shifted its position in yards during the first fifteen minutes of play as follows: + 50, − 5, + 12, − 15, − 3, + 30. Where was the ball at the end of the last play? (Distance towards the north goal is here regarded as +.)

37. The differences in readings of a thermometer that was read hourly from 7 A.M. until 7 P.M. were as follows: 3°, 4°, 7°, 10°, 12°, 15°, 10°, 5°, 0°, − 20°, − 15°, − 10°. How did the the temperature at 7 P.M. compare with that at 7 A.M.? If the temperature at 7 A.M. was + 20°, what was the temperature at each hour of the day?

38. One man has property worth \$1000, and his debts amount to \$1000. Another man who has no property at all owes \$100. Which of these two men is the better off financially?

39. If you consider profit or gain as being +, how must loss be considered? With this understanding what is the difference between a − gain and a loss? What does − \$5 profit mean? − \$10 loss? Now suppose a man bought a horse for \$150 and sold him for \$125. What did he gain? If the horse had been sold for \$160 what would have expressed the loss? What is the difference between a − loss and a gain?

40. A boy has \$25 in the bank, \$15 in his pocket, and two other boys together owe him \$10 while he himself owes \$15. How much is he really worth?

41. Let $a = -3, b = -2$, and show that $\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$. Show that this expression also holds for other values of a and b, positive as well as negative.

42. Find the value of $\frac{x^2 + 2xy + y^2}{x - y}$, for $x = -2$; $y = -1$.

Express the answers to the following in *algebraic language*, that is, in terms of some letter.

43. Express y dollars in cents. *Ans.* $100y$ cents.

44. Express in terms of y the interest on \$100 for 3 years at y %; the interest on y dollars at 6% for three years; the interest on \$100 for y years at 6%.

45. A man bought a farm for \$$x$ and sold it at a gain of $(x - 200)$ dollars. Express the selling price in terms of x.

46. A man bought two automobiles at $2x$ dollars apiece, and sold one at a profit of 25% and the other at a loss of 25%. Express the selling price of each in terms of x. If he had sold one at a profit of y% and the other at a loss of y%, what expressions would represent the two selling prices? If he had sold one at a profit of y dollars and the other at a loss of y dollars, what expressions would then represent the selling prices?

47. A rectangle is h feet long and $h - 10$ feet wide. Express in terms of h the area of the rectangle.

48. Express the perimeter of the rectangle in Example 47 in terms of h.

49. How much can a man save on a salary of e dollars a year if his expenses for half the year average \$75 a month and for the other half year, x dollars a month? (Express in terms of e and x.)

50. A horse valued at $\$x$ was sold at a loss of $y\%$. What was the selling price?

Ans. $\$x - \dfrac{\$xy}{100}.$

CHAPTER III

ADDITION AND SUBTRACTION

33. Signs of Aggregation are the parentheses ( ), brackets [ ], and braces { }. These signs are often used to inclose numbers. When so written the numbers inclosed are to be treated as a single number. For example, $(a + b)$ is read binomial a plus b, and means that we are to treat the sum of a and b as a single number.

34. Addition by Grouping Numbers. In adding a column or row of numbers we sometimes group the numbers in twos or threes. Thus, in adding 3, 7, 2, and 8, we observe that $3 + 7 = 10$, that $2 + 8 = 10$ and that $10 + 10 = 20$. In general, this means that $a + b + c + d = (a + b) + (c + d) = (a + c) + (b + d)$, etc.

In words this means that *the sum of several numbers is always the same in whatever way they may be grouped.*

35. Addition of Terms which have no Common Factor.

The sum of a and b is $a + b$.

The sum of $2a$, $-b$ and 4 is $2a - b + 4$.

The sum of $5n$ and $-6p$ is $5n - 6p$.

In general, to add terms which have no common factor, *bring down the terms in succession each with its own sign.*

36. Subtraction of Terms which have no Common Factor.

a taken from b leaves $b - a$. (a is the number to be subtracted.)

$-3c$ taken from $4d$ leaves $4d + 3c$. ($-3c$ is the number to be subtracted.

$-2x$ taken from $-y$ leaves $-y + 2x$ or $2x - y$. ($-2x$ is the number to be subtracted.)

In general, to subtract terms which have no common factor: *Change signs of terms to be subtracted and then bring down the terms in succession each with its own sign.*

EXERCISE

37. In Examples 1–2–3–4–5–6 write down the sum.

1. 5, $3a$, and $2x$.
2. ax^2, dy, and cx.
3. a, $-b$, $-c$, and d.
4. $3(a+b)$, $5(a-b)$ and $6(c+d)$.
5. $2mn$, $4m^2$, and $7(m-n)$.
6. $-a$, $+cd$, $-b$, and $-e$.
7. a taken from $-b$ = ?
8. c taken from $-d$ = ?
9. $-2f$ taken from $-3g$ = ?
10. $-4m$ taken from $-5p$ = ?

38. Addition of Terms which have a Common Factor.

Example 1. Add $5a$, $6a$, $3a$, and $7a$.
Solution.
$$5a + 6a + 3a + 7a = (5 + 6 + 3 + 7)a,$$
$$= 21a.$$

Example 2. Add $-5s$, $3s$, $7s$, and $-9s$.
Solution.
$$-5s + 3s + 7s + (-9s) = (-5 + 3 + 7 - 9)s$$
$$= -4s.$$

Example 3. Add $5(a-b)$, $6(a-b)$, and $-4(a-b)$.
Solution.
$$5(a-b) + 6(a-b) + (-4)(a-b) = (5 + 6 - 4)(a-b)$$
$$= 7(a-b).$$

In general, *to add terms which have a common factor, add the coefficients of the common factor and multiply the common factor by their sum.*

EXERCISE

Find the sum of:

1. $8x$, $7x$, $-6x$, and $-10x$.

2. $3a^2$, $-7a^2$, $-5a^2$, and $10a^2$.

3. $5a^2b$, $10a^2b$, $-8a^2b$, and $-a^2b$.

4. $-xy$, $3xy$, $8xy$, $-10xy$, and $7xy$.

5. $-\frac{1}{2}abc$, $\frac{3}{4}abc$, $\frac{5}{8}abc$, $-abc$.

6. $(x-y)$, $-4(x-y)$, $-7(x-y)$, and $15(x-y)$.

7. $5.4a^2b^2c^2$, $-2.8a^2b^2c^2$, $-8.2a^2b^2c^2$, and $1.7a^2b^2c^2$.

8. $\frac{1}{2}gt^2$, $\frac{3}{4}gt^2$, $-gt^2$.

9. $9s$, $4\frac{1}{2}s$, $-3\frac{1}{4}s$, $8s$, $-17s$.

10. $9(a+b)$, $-4(a+b)$, $(a+b)$, $6(a+b)$, $-3(a+b)$ and $2(a+b)$.

11. $6(R+H)$, $-4(R+H)$, and $-8(R+H)$.

12. $2(P-q)$, $-3(P-q)$, $-(P-q)$.

13. $2 \cdot$ angle A, $4 \cdot$ angle A, and $-3 \cdot$ angle A.

14. $6 \cdot 45°$, $7 \cdot 45°$, and $2 \cdot 45°$.

15. $7a(y)$, $6c(y)$, $-3b(y)$, y is common factor here.

16. $10(-2°)$, $4(-2°)$, $-3(-2°)$.

17. $a \cdot \$5$, $b \cdot \$5$, $-c \cdot \$5$.

18. $a(x-y)$, $-b(x-y)$, $-c(x-y)$.

19. $p(-m)$, $q(-m)$, $r(-m)$.

20. $a'(gt^2)$, $a''(gt^2)$, $a'''(gt^2)$.

39. Subtraction of Terms which have a Common Factor.

Example 1. Take $5a$ from $8a$.

Solution. $8a - 5a = (8-5)a$
$= 3a.$

Example 2. Take $-3a$ from $8a$.

Solution. $8a - (-3a) = 8a + 3a$, since subtracting $-3a$ means the same thing as adding $+3a$
$= (8+3)a$
$= 11a.$

Example 3. Take $3(x+y)$ from $-7(x+y)$.

Solution. $$-7(x+y)-3(x+y)=(-7-3)(x+y)$$
$$=-10(x+y).$$

In general, *to subtract terms which have a common factor, subtract the coefficients of the common factor and multiply their difference into the common factor.*

EXERCISE

1. From $8x^3$ take $-4x^3$. **2.** From $12a^2b$ take $16a^2b$.

3. From $-18t^2$ take $-12t^2$. **4.** From $-xy$ take $\frac{1}{2}xy$.

5. From $3.5mn$ take $-1.5mn$. **6.** From 0 take $2ab$.

7. From $(a+b)$ take $2(a+b)$. **8.** From $-(x-y)$ take $-3(x-y)$.

9. From $\frac{2}{3}(m+n)$ take $-\frac{1}{3}(m+n)$. **10.** From 0 take $2(x+y)$.

11. From $\frac{a+b}{2}$ take $\frac{a+b}{4}$.

12. From $3(-cd)$ take $.5(cd)$.

13. From the sum of $3(x-7)$, $-7(x-7)$, and $8(x-7)$ subtract the sum of $5(x-7)$ and $-2(x-7)$.

14. From the difference between $-9a^2bc$ and $4a^2bc$ take their sum. From their sum take their difference.

15. From $8 \cdot$ angle B take $3 \cdot$ angle B.

16. From $\frac{1}{2}gt^2$ take $-\frac{1}{16}gt^2$.

17. From $10 \cdot 30°$ take $3 \cdot 30°$.

18. From $a'gt^2$ take $a''gt^2$. (gt^2 is the common factor.)

19. From $a \cdot \$5$ take $b \cdot \$5$. **20.** From $-3(10°)$ take $-4(10°)$.

40. Similar Terms are those which are alike in their literal parts.

Thus, $4xy$, $5xy$, and $-6xy$ are similar terms.

Also, $6a^2bc$, a^2bc, and $-7a^2bc$ are similar terms.

41. The Addition and Subtraction of Polynomials.

Example 1. Find the sum of $a + 2ab + b$, $4a - 8ab + 4b$, and $9a + 18ab + 9b$.

Solution.

$$\begin{array}{rcrcr} a & + & 2ab & + & b \\ 4a & - & 8ab & + & 4b \\ 9a & + & 18ab & + & 9b \\ \hline 14a & + & 12ab & + & 14b \end{array}$$

Explanation. It is immaterial which column is added first. Hence, beginning on the left, the sum of $9a$, $4a$, and a is $14a$, which is brought down as the first term of the result. Next, the sum of $18ab$, $-8ab$, and $2ab$ is $+12ab$, which is brought down as the second term of the result. Finally, the sum of $9b$, $4b$, and b is $+14b$, which is brought down as the last term of the result.

Example 2. In like manner to subtract $c - 2cd + d$ from $4c - 4cd + d$, the expressions are set down as follows:

Solution.

$$\begin{array}{rcrcr} 4c & - & 4cd & + & d \\ c & - & 2cd & + & d \\ \hline 3c & - & 2cd & & \end{array}$$

Explanation. It is immaterial which column is subtracted first. Hence, beginning with the left-hand column and following the rule for the subtraction of similar terms or terms having a common factor, we have $4c - c$ or $3c$ for the first term of the result. In like manner, $-2cd$ is found to be the other term of the result.

42. Summary. The processes of addition and subtraction of polynomials, as illustrated in the foregoing examples, may be stated as follows:

(1) *Write similar terms in the same column.*

(2) *For addition of polynomials, add the terms in each column and write the results in succession with their proper signs.*

(3) *For subtraction of polynomials, change the sign of each term to be subtracted, add each of these terms whose signs have been changed to the other term in same column and write the results in succession with their proper signs.*

43. Checking Results. A check on an operation is another operation to test the correctness of the first.

To illustrate: Let $a = b = 1$, in the foregoing example in addition. The solution and its check are as follows:

$$
\begin{array}{lrcrcr}
 & \textbf{Solution} & & \textbf{Check} & & \\
 & a + 2ab + b & = & 1 + 2 + 1 & = & +4 \\
 & 4a - 8ab + 4b & = & 4 - 8 + 4 & = & 0 \\
 & \underline{9a + 18ab + 9b} & = & \underline{9 + 18 + 9} & \underline{=} & \underline{+36} \\
\text{Sum,} & 14a + 12ab + 14b & = & & & +40 \\
\text{But} & 14a + 12ab + 14b & = & 14 + 12 + 14 & = & +40
\end{array}
$$

Therefore, since the numerical value of the sum is $+40$ and the sum of the numerical values of the three polynomials is also $+40$, we conclude that the result is correct.

Similarly, in checking the result of the foregoing example in subtraction, the solution and its check appear as follows: Let $c = d = 1$.

$$
\begin{array}{lrcrcr}
 & \textbf{Solution} & & \textbf{Check} & & \\
 & 4c - 4cd + d & = & 4 - 4 + 1 & = & 1 \\
 & \underline{c - 2cd + d} & = & \underline{1 - 2 + 1} & \underline{=} & \underline{0} \\
\text{Difference} & 3c - 2cd & = & & & 1 \\
\text{But} & 3c - 2cd & = & 3 - 2 & = & 1
\end{array}
$$

Therefore, since the numerical value of the difference is 1, and the difference of the numerical values of the two polynomials is also 1, we conclude that the result is correct.

Such a numerical check as the foregoing will usually, though not always, detect errors. A more reliable check can be obtained by the substitution of a different number for each letter.

EXERCISE

(Check results to first ten exercises.)

44. 1. Add: $4\,a^3b + 4\,a^2b - 2\,ac^2$, $a^3b - a^2b + ac^2$, $-5\,a^3b + 7\,a^2b + 9\,ac^2$ and $a^2b + 2\,ac^2$.

2. $-9\,t^2 + 3\,t - 8\,t^3$ taken from $16\,t^2 - t + 4\,t^3$ leaves ?

3. Add: $x^4 + 3\,x^3 + 2\,x^2 + x + 7$, $2\,x^4 - 3\,x^3 - 6\,x^2 + 5\,x - 4$, and $-7\,x^4 + 6\,x^3 - 11\,x^2 + 1$.

4. Add: $\frac{1}{2}\,gt^2$, $\frac{1}{4}\,gt^2$, $\frac{3}{4}\,gt^2$, $7\,at^2$, $-6\,at^2$, $\frac{1}{4}\,gt^2$, and $-\frac{1}{4}\,at^2$.

5. Add: $4\,a^2b - 3\,xy + 2\,cd$ and $a^2b - 3\,xy - 4\,cd$.

6. From the sum of $16\,abc + 7\,b^2c^2 - 9\,acd$ and $-12\,abc + 15\,b^2c^2 + 12\,acd$, take the sum of $abc - b^2c^2 + 3\,acd$ and $10\,abc + 14\,b^2c^2 - 15\,acd$.

Add the following polynomials:

7. $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$, $a^3b + 3\,a^2b^2 + 3\,ab^3 + b^4$, and $a^2b^2 + 2\,a^4 - b^4 - 7\,a^3b$.

8. $7(a^2 + b^2) + 8(a^2 - b^2) - c^2$, $-10(a^2 + b^2) + 5(a^2 - b^2) + 4\,c^2$ and $2(a^2 + b^2) - 10(a^2 - b^2) - 2\,c^2$.

9. $\frac{1}{2}\,a^2c^2 + \frac{2}{3}\,a^2b^2 - \frac{1}{4}\,b^2c^2$, $\frac{1}{4}\,a^2c^2 - \frac{1}{2}\,a^2b^2 - \frac{3}{4}\,b^2c^2$, and $\frac{1}{4}\,a^2c^2 - a^2b^2 + \frac{1}{2}\,b^2c^2$.

Subtract:

10. $x^2 + 2\,xy + y^2$ from $x^2 - 2\,xy + y^2$.

11. The sum of $a^2 + 4\,ab + b^2$ and $a^2 - 2\,ab + b^2$ from $a^2 - 4\,ab + b^2$.

12. $4(x - 1) + 3(x + 2) + 5(x - 4) - (x - 2)$ from $(x - 1) - 3(x + 2) + 2(x - 4) + 2(x - 2)$.

13. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ from $-2\,x^3y + 4\,x^2y^2 - 2\,xy^3$.

Simplify the following expressions by combining the similar terms:

14. $a - b + 2c + d + 3a - 2c - d - 2b + 5a - 2d + 3c$.

15. $2xy + y^2 + x^2 - 4xy + 3x^2 - 2xy$.

16. $7(a - x) + (a + y) - 4(a - x) - 7(a + y) + (a - x)$.

17. $11a^2b + 7c^2x - 8a^2y + a^2b - 4c^2x + a^2y$.

18. $10(u + v) + 3(u - v) - 4(u - v) + (u + v)$.

Subtract:

19. $x^2 + y^2$ from $-2xy$.

20. $-2xy$ from $x^2 + y^2$.

21. $m + n + p + q$ from $m - n$.

22. $a - b - c - d - e$ from a.

23. $x + y - z - 1$ from 3.

24. The temperature at 7 A.M. was 41° and rose 28° during the forenoon; during the afternoon it fell 34°. What was the temperature at the close of the afternoon? (Always consider rising temperatures as positive unless otherwise specified.)

25. A thermometer which stood at $a°$ at 7 A.M. rose $b°$ during the forenoon and fell $c°$ during the evening. What expression would then indicate the temperature?

26. The assets of a firm were appraised as follows:

Real estate	valued at x dollars,
Merchandise	valued at y dollars,
Accounts receivable	valued at z dollars.

The firm's liabilities consisted of:

Accounts payable to the amount of a dollars;
Notes and mortgages amounting to b dollars.

How much was the firm worth?

27. A commercial traveler starting from Chicago in the morning made the following trips in the order stated along an East and West route during the day:

a miles west; b miles east; c miles west; d miles east.

How far was he from Chicago at the end of the day?

28. The temperature was 20° at 7 A.M.; by 11 A.M. it had risen $a°$ and by 1 P.M. a further rise of $b°$ had been recorded. It then began to fall and by 6 P.M. a fall of $c°$ had occurred. What was the temperature at 6 P.M.?

29. A miner was 1000 feet below the surface of the ground. Regarding distance upward as positive, how far below the surface was he (*a*) after being lifted b feet? (*b*) after being lifted b feet and then lowered c feet?

30. A balloon, 2000 feet above the earth, ascended a feet, then descended $(1000 + b)$ feet, then rose c feet, and finally descended e feet, where it remained stationary. Considering distances upward as positive, what was its final elevation?

45. Meaning of Signs of Aggregation as used in Addition and Subtraction.

For example: $4a + (3b - c)$ means $4a + 3b - c$,
and, $x + y - q$ means $x + (y - q)$; also $(x + y) - q$.
But, $4a - (3b - c)$ * means $4a - 3b + c$,
and, $x + y - q$ means $x - (-y + q)$;
also, $-(-x - y) - q$.

From the foregoing it will be seen that the parentheses, or any other sign of aggregation, may be removed or inserted at any time when preceded by the + sign without any other change. But when preceded by the − sign, the whole quantity within the parentheses is to be subtracted; hence, *we must change the sign of each term inside the parentheses if we remove them.* Likewise, if we insert the parentheses preceded by the − sign, *we must change the sign of each term that is inclosed by the parentheses.*

When more than one of the signs of aggregation occur, as they usually do, one within the other, it is best to remove the innermost first.

---

* Observe that the sign of $3b$ in the binomial $(3b - c)$ is +.

Thus in

$$[a + b - \{c + d - (e - f)\}],$$

first remove the parentheses,

$$[a + b - \{c + d - e + f\}],$$

next remove the braces,

$$[a + b - c - d + e - f],$$

and finally remove the brackets,

$$a + b - c - d + e - f.$$

EXERCISE

In the following examples, remove the parentheses, etc., and simplify wherever possible by combining similar terms:

1. $2a - 3[b + c + \{a - b - (a - c)\}]$.
2. $7(a - b) - 3(a + b) - 5\{a + b - (a - b)\}$.
3. $10(mn - pq) + 4\{mn - pq - (mn - 2pq)\}$.
4. $-[6a^3 - b^3 + \{a^3 - (b^3 - a^3 + 3 \cdot a^3 - b^3)\}]$.
5. $2a^2b^2 - [\frac{3}{4}a^2b^2 - \{(\frac{1}{4}a^2b^2) - \frac{1}{4}a^2b^2 - (-4a^2b^2)\}]$.
6. $12 - \{-8 + a - b + c - d(-3)\}$.
7. $5b - (8 - 4b - 10c) + (-5b + 3c) - (8c + 3b - 9)$.
8. $10 - (-5 - a + b - c) + 12c - (7 - 2a + b) - (-8a - 5 + 6c)$.
9. $8(4a - 3k + 7L) - 6(-2a + k - L) - 20(a + k + L - a)$.
10. $-2(a - b - c) + 3(-a + b + c) - (5a - 6b - 7c)$.

In the following check the answers by assuming that $a = b = c = 1$.

11. $-(abc - a^2bc) + a^2 - (-b^3 - c^3) - 2abc - (-5a^2bc)$.
12. $-(a^2 - b^2 + c^2) - 4(b^2 - c^2 - a^2) - 3(c^2 - b^2 - a^2)$.

13. $-a(b-c)+4\,ab-b(a-c)-c(-a-b)$.

14. $\frac{1}{2}\,a-(b-c)+b-(a-c)-c(b-a)$.

Insert parentheses preceded by − sign in the following expressions, inclosing in each the last two terms:

15. $a-b-c-d$.

16. $b-a-c-d$.

17. $c-a-b-d$.

18. $d-a-b-c$.

19. $x^2-y^2-x^2+p^2$.

20. $-x^2+y^2+z^2-q^2$.

21. $2\,a+b-c+d$.

22. $m-2\,mn-2\,pq+2\,rs$.

MISCELLANEOUS EXERCISES FOR REVIEW

46. 1. From $x^3+3\,x^2+3\,x+1$ take $x^3-3\,x^2+3\,x-1$. Check by putting $x=1$.

2. Add $3\,x^2y^2-10\,y^2$, $5\,y^2-x^2y^2$, $7\,x^2y^2-6\,y^2$, $3\,x^2+4\,x^2y^2+2\,y^2$. Check with $x=2$, $y=1$.

3. Take $-\frac{2a^2}{3}+a-\frac{1}{2}$ from $\frac{2a^2}{3}-\frac{5\,a}{2}-1$. Check with $a=1$.

4. What expression must be added to $7\,y^2-1$ to produce $5\,y^2-7\,y+2$?

5. Take $4\,n^2-5\,n+2$ from 1 and add $5\,n^2-6\,n+3$ to the result.

Simplify the following:

6. $5(a-b)-7(a-b)-(a-b)+6(a-b)$.

7. $3(r-s)^2+7(r-s)^2-2(r-s)^2$.

8. $2\,a-2(3\,a-b)-3(b+2\,a)$.

9. $b-5\,c-a-3(a-c)+2\,c-(-2\,b-c+a)$.

10. $9\,m+(-4\,m+6\,n)-(3\,m-n)$.

11. $3\,x^2-2\,xy+7\,y^2+5\,x^2+9\,xy-10\,y^2+8\,x^2-6\,xy-4\,y^2$.

12. $(ax-2\,by+3\,cz)+(by-2\,cz+3\,ax)+(cz-2\,ax-3\,by)$.

13. $(6\,x^2y^2+3\,x-7\,xy)-(3\,xy-4\,y-7\,x^2y^2+2\,x^2y^2)$
$+(x-x^2y^2+xy)-(4\,x+4y)$.

14. $mx - 8y + (6a - mx) - 2a.$

15. $4a^2 - 6a + 4 - 3a^2 + a^2 + 1 \cdot 5a - 2 - 3 \cdot 4a - 3 \cdot 75.$

In the following collect the terms with respect to x and y; that is, write the expressions with polynomial coefficients:

16. $ax + by + cx + dy - ex.$ **17.** $2ax + 3bx - 4x.$

18. $my - ny + px - qx - ry + tx.$

19. $5(x^2 - y^2) - a(x^2 - y^2) + 3b(x^2 - y^2) - c(x^2 - y^2).$

20. $3(a - x) - 4c(a - x).$

21. $5y - 3acx - 5cdy - 4abx - 3cdy + 2cx - 4y - 5ax.$

In the following collect the coefficients of x, x^2, and x^3:

22. $ax^3 + bx^2 - ex - dx + ex^2 + gx^3 - kx^2.$

23. $-x^3 - x^2 - x + ax^3 + bx + cx^2 - 2bx^3 - 2cx^2 - 2ax^3.$

24. The sum of two numbers is $3b + c$; and their difference is $b - c$. What are the numbers?

25. The difference between two consecutive numbers, one of which is k, is $k - 1$. What are the numbers?

26. The sum of two consecutive numbers is $2k + 1$. What are the numbers?

27. Write as a binomial (*a*) $a - b + c - d.$ (*b*) $a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$

28. Write $a - b + c - d + e - f + g - h$ (*a*) as a succession of binomials each of which is preceded by the + sign. (*b*) as a succession of binomials each of which is preceded by the − sign.

Express the following statements as equations:

29. The sum of x and five is twelve. **30.** x is five more than a.

31. x exceeds y by four. **32.** x is four less than y.

33. The sum of x and five is the same as the sum of y and ten.

34. Seven added to x gives the same result as x taken from twenty.

35. Ten taken from three times x leaves the same number as x added to thirty.

36. Six times a plus three times b gives twenty.

37. Seven times s square plus four times b cube gives c to the fourth power.

38. a cube plus b cube less three times the product of a, b, and c gives zero.

39. The sum of the squares of a and b gives c.

40. The difference of the cubes of a and b gives c cube.

41. Four times a exceeds eighteen by six.

Verify your answers to the two following by using a thermometer:

42. How many degrees and in what direction must the temperature change to vary from 15° below zero to 45° above zero? $(-15^\circ + ? = 45^\circ)$.

43. How many degrees and in what direction must the temperature change to vary from 50° above zero to 12° below zero? $(50^\circ - ? = -12^\circ)$.

44. How many dollars must be added to a loss of $25 in order to change this loss to a gain of $50?

45. What sum must be added to a gain of $10 in order that the gain be changed to a loss of $10?

46. An elevator starting from the ground floor goes up 60 feet, then down 15 feet, then up again 30 feet, and then down 45 feet.

(*a*) How many feet above the ground floor was it at the end of the last movement?

(*b*) Considering distance up as positive, what expression will represent the total distance passed through by the elevator?

47. An elevator starting from the ground floor goes up a feet, then down b feet, then up c feet, and finally down d feet. Considering distance up as positive, how many feet from the ground floor was it at the end of the last movement?

48. A man left an estate valued at $10,000 to be divided among his three sons as follows: To the youngest son a dollars, to the next older b dollars more than to the youngest, and to the oldest c dollars more than twice the share of the youngest. Express these statements in the form of an equation.

CHAPTER IV

LINEAR EQUATIONS

(a) Solution

47. An Equation is an expression of equality between two numbers.

Thus, $5 = 3 + 2$, $3x = 12$, $5n = 7 - 2n$, and $3x^2 + 2x + 5 = 0$ are equations.

That part of an equation to the left of the equality sign is called the first member; and that to the right, the second member.

48. A Numerical Equation is an equation whose coefficients and known terms are represented by the Hindu numerals.

Thus, in the equation $7x = 3x + 8$, x represents the unknown number while the coefficients 7 and 3 and the known number 8 are all represented by the Hindu numerals.

On the other hand, $ax = bx + c$ and $7x = 3x + c$ are literal equations. These last will be considered later.

49. The Degree of an Equation * containing a single unknown number is determined by the highest exponent of the letter representing the unknown number.

For example: $5x + 4 = 3 - x$ is an equation of the first degree in x, since the highest exponent of x is 1.

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* All powers here considered are supposed to be integers.

$3x^2 + x = 7$ is an equation of the second degree in x, since the highest exponent of x is 2.

Similarly, $x^3 - 5x = x^2 + 7$ is of the third degree in x, and $ax^n + bx^{n-1} + \dots + e = 0$ is an equation of the nth degree in x.

50. A Conditional Equation is an equation that holds for particular values only of the letter representing the unknown number.

Thus $5x = 10$ is true only when $x = 2$.

On the other hand $5x - 3x = 2x$ is true when $x = 1$, or when $x = 2$; and equally as well for any other value of x. Such equations as these last are *identical equations*, or *identities*. Whenever the term "equation" is used without any qualification, it means a conditional equation.

51. A Linear Equation in one letter contains the unknown number in the first degree only.

Thus $10x - 3 = 0$ and $ax + b = 0$ are linear equations in x.

Also, an equation in two letters is linear if each of these letters occurs only to the first power and both do not occur in the same term as a product.

Thus $6x - 3y = 10$ and $ax + by = c$ are linear equations in x and y.

Also, $\dfrac{6x}{y} = 50$ is linear, since it really becomes $6x = 50y$.

But $16xy = 30$ and $6x = \dfrac{10}{y}$ are not linear in x and y.

If a linear equation has (*a*) integral coefficients; or (*b*) fractional coefficients, but does not have the letter representing the unknown number in the denominator, it is called an *integral* equation.

Thus $3x + 4 = 5$ and $\frac{x}{2} + \frac{x}{3} = 6$ are integral equations.

On the other hand $\frac{x}{2} + \frac{5}{x} = 2$ is not integral, but fractional.

52. Illustrative Solutions of Linear Equations.

Example 1. Solve for x, $10x + 9 = 30 + 3x$.

Solution. $10x - 3x = 30 - 9$, subtracting $3x$ and 9 from both sides.

(Axiom 2: Equals subtracted from . . .)

$7x = 21$, collecting terms.

Therefore, $x = 3$, dividing both sides by 7.

(Axiom 4: Equals divided by equals . . .)

Check. $10 \cdot 3 + 9 = 30 + 3 \cdot 3$,

$30 + 9 = 30 + 9$,

$39 = 39$.

Example 2. Solve: $\frac{x}{2} - \frac{3}{4} = x - \frac{2}{3}$.

Solution. $6x - 9 = 12x - 8$, multiplying both sides by 12.

(Axiom 3: Equals multiplied . . .)

$6x - 12x = 9 - 8$, subtracting $12x$ from both sides and adding 9 to both sides.

$-6x = 1$, collecting terms.

Therefore, $x = -\frac{1}{6}$, dividing both sides by -6. (Axiom 4.)

Check. $\frac{1}{2}(-\frac{1}{6}) - \frac{3}{4} = -\frac{1}{6} - \frac{2}{3}$.

$-\frac{1}{12} - \frac{3}{4} = -\frac{1}{6} - \frac{2}{3}$.

$-\frac{10}{12} = -\frac{10}{12}$.

53. Summary. From the foregoing solutions it will be observed that:

(1) We may perform all or any one of the following operations upon an equation:

(*a*) *Add the same number to both sides.*

(*b*) *Subtract the same number from both sides.*
(*c*) *Multiply both sides by the same number.*
(*d*) *Divide both sides by the same number.*

(2) A linear equation in one letter may be solved as follows:

(*a*) *If the equation has fractional coefficients, multiply both sides by such number as will remove the fractions.*

(*b*) *By the addition and subtraction of proper terms transform the equation into one in which all the known numbers are in one member and all the terms containing the unknown number are in the other member.*

(*c*) *Collect the terms in each member of the equation.*

(*d*) *Divide both sides by the coefficient of the letter representing the unknown number.*

(*e*) *Check the result obtained by substituting it for the letter representing the unknown number in the equation. If the equation reduces to an identity the result is said to satisfy the equation.* A number that thus satisfies an equation is called a *root* of the equation.

EXERCISE

54. Solve each of the following equations, checking results as shown in the foregoing illustrative examples:

1. $10y + 7 = 37.$

2. $5n - 4 = n + 12.$

3. $10 - t = 20 - 5t.$

4. $1 - 10x = 4x - 12.$

5. $0 = 10x - 4x - 12.$

6. $\frac{2y}{5} - 8 = 2 - y + \frac{y}{2}.$

7. $7r - 12 = 3r - \frac{4}{2}.$

8. $v + 2v - 6 = 3v - \frac{v}{2} + 10.$

9. $27m + 189 = 12m - 36.$ **10.** $-x - 5 = x - 15.$

11. $5(x - 3) = 3(x - 3) - 10.$

12. $\frac{2}{3}(x - 6) + \frac{1}{2} = x - (3 - 2x).$

13. $3(x - 5) - 2(x - 4) = 0.$ **14.** $\frac{6}{5} - \frac{y}{2} = 3y - 15.$

15. $\frac{3}{5}(y + 7) + \frac{1}{3}(y - 4) = -\frac{3}{4}.$

16. $7k - \frac{k}{4} = -k - 7.$ **17.** $9c - \frac{2}{3} = \frac{1}{2}.$

18. $2k + 7 = 4k + 5.$ **19.** $10t - \frac{1}{2} = -10t + \frac{1}{4}.$

20. $3x - 2 = -4x + 12.$

55. Illustrative Solutions of Problems.

Problem 1. Divide a line $11\frac{1}{2}$ inches long into three parts, a, b, and c, so that c shall be 1 inch longer than b and twice as long as a.

Note, first, that this problem contains three unknown numbers: the lengths of the three parts of the line which are designated a, b, and c.

Second, that the problem contains three conditional statements:

1. Sum of all the parts, a, b, and c must equal $11\frac{1}{2}$.
2. The longest part c must be 1 inch longer than b;
3. The longest part c must be twice as long as a.

Solution.

Let s = the length of a, the shortest part;
then $2s$ = the length of c (3d condition)
and $2s - 1$ = the length of b (2d condition).
Then $s + 2s + (2s - 1) = 11\frac{1}{2}$ (1st condition).
$s + 2s + 2s = 11\frac{1}{2} + 1$ (adding 1 to both sides of the equation).
$5s = 12\frac{1}{2}$, collecting terms.

Therefore, $s = 2\frac{1}{2}$, dividing both sides by 5,
and $2s = 2(2\frac{1}{2}) = 5$,
and $2s - 1 = 5 - 1 = 4$.

Hence, a is $2\frac{1}{2}$ inches long; c is 5 inches long; and b is 4 inches long.

Check. $5 - 1 = 4$; $5 = 2(2\frac{1}{2})$, and $5 + 4 + 2\frac{1}{2} = 11\frac{1}{2}$.

Problem 2. A is 29 years old and his son is 8 years old. In how many years will A's age be four times the age of his son?

Note, first, that this problem involves three unknown numbers:

1. The number of years that must elapse before A's age will be equal to four times his son's age;
2. A's age at that time;
3. The son's age at that time.

Note also that the problem contains three conditional statements:

1. A's age now;
2. The son's age now;
3. A's age at a certain time must equal four times the son's age.

Solution.

Let $x =$ the required number of years;
then $29 + x =$ A's age at expiration of the required time; 1st condition
and $8 + x =$ son's age at expiration of required time; 2d condition.
Then $29 + x = 4(8 + x)$, third condition.
$29 + x = 32 + 4x$, removing the parentheses.
$x - 4x = 32 - 29$, subtracting $4x$ and 29 from both sides.
$-3x = 3$, collecting the terms.
$x = -1$, dividing both sides by -3.

Therefore, in -1 year, that is, one year ago, A's age was equal to four times the age of his son.

Check. $29 - 1 = 4(8 - 1)$,
$28 = 28$.

56. Summary. The solution of problems in algebra, as observed in the foregoing illustrative problems, may be summed up as follows:

(1) *In every problem there will be found certain unknown numbers. Represent one of these by some letter.*

(2) *In every problem there will be found also certain conditional statements; in fact, just as many conditional statements as unknown numbers. Having represented one of the unknowns by some letter, express each of the other unknowns in terms of this letter, each time using one of the conditional statements that remain. This will leave one conditional statement unused.*

(3) *Form the equation from this remaining conditional statement.*

(4) *Solve the equation by methods of* § 53.

PROBLEMS

57. Problems Relating to Numbers.

1. The sum of two numbers is 246 and their difference is 64. Find the numbers.

2. The sum of two numbers is s and their difference is d. What are the numbers?

3. Using the results obtained in Example 2 as formulas, determine directly by substitution two numbers:

(*a*) Whose sum is 45 and whose difference is 15.

(*b*) Whose sum is -10 and whose difference is -4.

4. The sum of four consecutive numbers is 18 and the last is twice the first. What are the numbers?

5. Divide 24 into three parts such that the first shall be half of the second and the second half of the third.

6. Divide 30 into three parts such that the second part shall be three times the first, and the last part half of the sum of the other two.

7. A number diminished by 6 gives the same result as the number divided by 4. What is the number?

8. Find five consecutive numbers such that the last is twice the first.

Problems Relating to Measurements.

9. It is required to divide a board 60 inches long into five parts such that the four longer parts shall be 1″, 2″, 3″, and 4″ longer respectively than the shortest part. Find the lengths of the different parts.

10. A fence 4 feet high is made out of 5″ boards running lengthwise. The number of boards necessary to build the fence up to the required height is 6, and they are so placed as to leave open spaces between them. If each of these open spaces, counting from the bottom upwards, is half of the one next above it, what must be the distances between the boards; that is, what will be the width of the open spaces?

11. The combined length of the Panama and Suez canals is 149 miles, and the second is 2 miles more than twice the length of the first. What is the length of each?

12. The combined length of the four longest rivers in the world, the Nile, the Amazon, the Mississippi, and the Missouri, is 12,960 miles. The Amazon is 100 miles shorter than the Nile and 200 miles longer than the Missouri; and the last-named is 60 miles shorter than the Mississippi. Find the length of each river.

13. The distance from New York to Denver is twice the distance from New York to Chicago and three-fifths of the distance from New York to San Francisco. If the combined distances, New York to Chicago, New York to Denver, and New York to San Francisco equal 6175 miles, find the distance from New York to Denver.

14. The estimated area of the coal lands of the United States is one and a half times the entire area of the state of Texas and twice the entire area of France. The sum of all three areas is 866,671 square miles. What is the area of the coal lands?

15. The length of a rectangle is 7 feet greater than twice its width. If its perimeter is 104 feet, what are its dimensions?

16. The length of a single tennis court is 51 feet greater than its width. If 9 feet be added to its width, the length remaining as before, the court will be a double court whose perimeter is $\frac{38}{35}$ of the perimeter of the single court. Find the dimensions of each court.

Problems Relating to Ages.

17. A man whose age is 36 has a son whose age is 11. In how many years will the father's age be twice the son's age?

18. Four years ago John's age was one-fourth of Henry's; and in five years John will be one-half as old as Henry. What is the age of each?

19. The sum of the ages of A, B, and C is 105 years. A is twice as old as C and B's age is one-third of the sum of the ages of A and C. How old is each?

Problems Relating to Motion.

20. A boy swimming at the rate of three miles an hour in quiet water strikes out into a current running at the rate of 2 miles an hour. How far can he swim going with the current in one hour? Against the current in one hour? How long would it take him to swim 10 miles down stream and back again?

21. A boy who can swim at the rate of 4 miles an hour in quiet water swims down stream 12 miles and back in 8 hours. What was the rate of the current?

22. A motor boat and a sail boat run over a course 30 miles in length, the motor boat with an average speed of 20 miles per hour, and the sail boat with an average speed of 8 miles an hour. Starting out together and following the same course, how far from the starting point will the sail boat be when it meets the motor boat returning?

23. A freighter running from Duluth to Buffalo averages 8 miles an hour. When 120 miles out from Duluth it is followed by the steamer *Northwest*, which averages 24 miles an hour. How far from Duluth will the *Northwest* pass the freighter?

24. Two trans-continental trains leave Kansas City at 11 A.M. and at 9 P.M. respectively. The first is scheduled to make 30 miles an hour and the second 40 miles an hour. How far from Kansas City will the second train overtake the first?

Miscellaneous Problems.

25. Divide $450 among three boys so that two shall have equal sums and the third $30 more than three times the amount received by each of the other two.

26. A man gave to his three daughters $1000, $1500, and $2000 respectively, with the understanding that at his death the estate should be so divided as to equalize the amounts received by all. If the estate was appraised at $22,000 how much of this did each daughter receive?

27. The distance from New York to San Francisco via Panama is 400 miles greater than the distance from San Francisco to Manila, and 2000 miles greater than the shortest all-rail route from New York to San Francisco. The sum of all three distances is 13,350 miles. Find the distance from New York to San Francisco via Panama.

28. The number of students attending the universities and colleges of the United States in 1912 was double the number in attendance for the year 1902 and one hundred fifty per cent greater than the attendance for 1892. The total attendance for the three years mentioned was approximately 682,000. What was the approximate attendance in 1892?

29. In four of the largest university and college libraries of the United States there are 3,000,000 volumes, June, 1912. The largest of these libraries contains twice as many volumes as the smallest; and 350,000 more volumes than the other two, which are equal in number of volumes contained. How many volumes are there in each of these libraries?

30. For the year 1911 the city of New York expended for the support of its public schools an amount equal to twice the sum expended by the whole state of Michigan, and three times the amount expended for the same purpose by the city of Chicago. Total expenditures of all three being \$66,000,000, what was the amount expended by each for the maintenance of its public school system?

31. The diameter of the sun is three and a half times the distance from the earth to the moon, and 400 times the diameter of the moon. The sum of all three distances, diameter of the sun, diameter of the moon, and distance from the earth to the moon, is 1,082,100 miles. Find the diameter of the sun.

Problems Illustrating the Use of the Equation in Geometry.

In geometry an equilateral triangle is defined as a triangle all three of whose sides are equal; an isosceles triangle, as a triangle two of whose sides and angles are equal; and a right triangle, as a triangle one of whose angles is a right angle; that is, an angle of 90°. The sides forming a right angle are perpendicular to each other. The angles of a triangle are usually designated by capital letters A, B, and C; the sides opposite by small letters a, b, and c respectively. Unless otherwise stated, such will be the significance of these letters in the exercises that follow.

32. Triangle ABC (Fig. 11) is an isosceles triangle whose base b is $1\frac{42}{100}$ times each of the equal sides, a and c; and the sum of whose sides is 12. What are the sides?

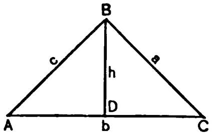

FIG. 11

33. In the same triangle ABC (Fig. 11), if angle B is a right angle, and the sum of all the angles, A, B, and C is 2 right angles or 180°, how many degrees in angle A?

34. Using the triangle ABC (Fig. 11) again, the altitude h bisects the angle B, and is perpendicular to the base b. How many degrees in each angle of triangle ABD? Of triangle CBD?

35. In right triangle *ACB* (Fig. 12), a is $\frac{3}{4}$ of b and b is $\frac{4}{5}$ of c. If the sum of a, b, and c is 24, what is the length of each side?

36. Using the same triangle, *ACB* (Fig. 12), if angle A is $\frac{2}{3}$ of angle B, C being a right angle, how many degrees are there in each angle?

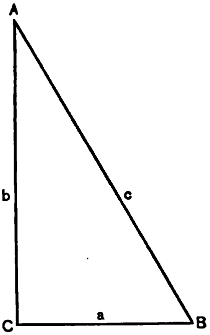

FIG. 12

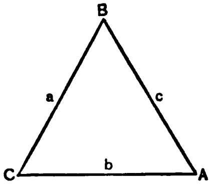

FIG. 13

37. What will be the value of each angle and each side of an equilateral triangle (Fig. 13) whose perimeter is equal to that of the triangle in Example 33? In Example 35?

38. In triangle *ABC* (Fig. 14), angle A is $\frac{3}{5}$ of angle C, and angle C is 20° less than angle B. How many degrees in each angle?

39. Using triangle *ABC* (Fig. 14) again, a is 2 units shorter than c, and b is 2 units longer than c. If the sum of a, b, and c is 31, what is the length of each side?

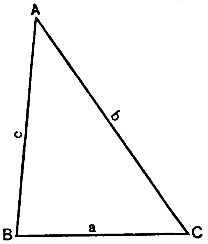

FIG. 14

40. Using the triangle *ABC* (Fig. 14), A is $\frac{2}{3}$ of C and C is $\frac{3}{4}$ of B. The sum of A, B, and C being 180°, how many degrees are there in each angle?

(b) Plot

58. Location of Places on the Earth. With a map of the earth before you let us suppose you are told that:

(1) A city is in 40° N. latitude. From this fact alone would you be able to give the exact location of the city? Why?

(2) A city is in 80° W. longitude. From this fact alone would you be able to give the exact location of the city? Why?

(3) A city is in 40° N. latitude and 80° W. longitude. Since the city lies at the intersection of two lines, the 40th parallel of North latitude and the 80th meridian of West longitude, you could now fix its exact location.

From the foregoing it will be observed that, in order to give the exact location of a city, there must be given (1) two fixed lines of reference, a standard meridian (usually that of Greenwich) and the equator; (2) the distance in degrees of the place from each of these lines of reference; (3) the direction of the place from each of these lines of reference.

59. Location of Points (on squared paper).

Let us assume that (1) XX' and YY' are two straight lines intersecting at right angles at 0. (Fig. 15.)

(2) A side of one of the small squares is $\frac{1}{10}$ of an inch in length and is to be taken as the unit of distance. Then the number 2 will correspond to a distance twice that of the unit, the number 3 to a distance three times that of the unit, etc.

(3) Distances are to be measured along or parallel to XX' and YY'.

(4) Distances to the right of YY' are positive; distances to the left of YY' will then be negative.

(5) Distances above XX' are positive; distances below XX' will then be negative.

(6) In measuring the distances to any point, the distance along or parallel to the line XX' shall be measured first and that along or parallel to YY', second.

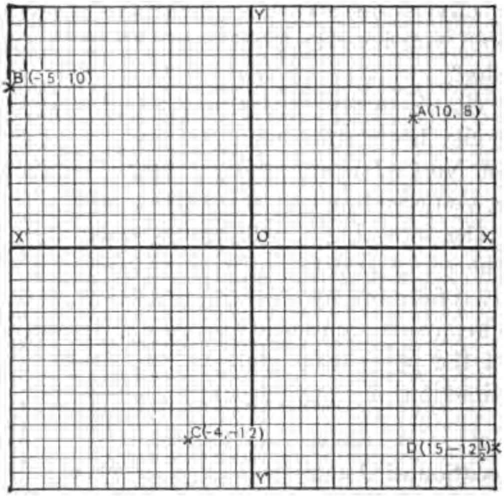

FIG. 15

With the foregoing assumptions in mind point A will be found to lie 10 units to the right of YY' and 8 units above XX'. A is designated as the point (10, 8). (Observe here that the number first written, 10, is the distance of the point from YY', measured along XX'; and the second number, 8, is the distance from XX', measured along YY'.) Point B will be found to lie 15 units to the left of YY' and 10 units above XX', and is designated as the point $(-15, +10)$. Point C will be found to lie 4 units to the left of YY' and 12 units below XX' and is designated as the point $(-4, -12)$. Point D will be found to lie

15 units to the right of YY' and $12\frac{1}{2}$ units below XX' and is designated as the point $(15, -12\frac{1}{2})$.

Since every one of the four points A, B, C, and D lies in the surface of the paper and corresponds to a *pair of numbers* and since the number of such points lying in the surface of the paper that may be thus designated may be increased indefinitely, it follows that *every point in the surface of the paper corresponds to a pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

60. Coordinates. The lines XX' and YY' as used in the preceding paragraph are called *axes*; and their intersection, designated 0, is called the *origin*. The distances of a point from the axes are called the *coordinates* of the point. Thus, 10 and 8 are the coordinates of A; -15 and 10 are the coordinates of B in the preceding paragraph. The coordinates of the origin are (0, 0). The coordinate first given is called the *abscissa*, and that given second, the *ordinate* of the point.

Thus 10 is the abscissa and 8 the ordinate of A in the foregoing discussion.

EXERCISE

(Use cross-section paper)

61. Locate the following points:

1. $(6, 3)$; $(4, -5)$.
2. $(-7, -2)$; $(-5, 4)$.

Locate the following points and using ruler test them to see if a straight line may be drawn through any three of them:

3. $(-1, -1)$; $(-1, -2)$; and $(-1, 4)$.
4. $(-2, -2)$; $(1, -1)$; and $(-1, 0)$.
5. $(0, -1)$; $(0, -2)$; $(2, 2)$; and $(-2, 2)$.

Construct the following lines:

6. A line such that the abscissa of every point on it is -3.
7. A line such that the abscissa of every point on it is 2.
8. A line such that the abscissa of every point on it is 6.
9. A line such that the ordinate of every point on it is 3.

62. The Graph of a Linear Equation in Two Letters. The equation $x + y = 10$ is satisfied if $x = 1$ and $y = 9$; or if $x = 2$ and $y = 8$; or if $x = -1$ and $y = 11$, and so on indefinitely for any pair of values whose sum is 10. It should be noted that any change or variation in the value of one of these numbers is accompanied by a change in the value of the other. Two unknown numbers standing in such relation to each other as that indicated by the equation $x + y = 10$ are called *variables*.

Tabulating some of the pairs of values that satisfy the equation $x + y = 10$, as at the right, we shall consider each of these pairs of values as the coordinates of a point, the x-value being the abscissa and the y-value the ordinate.

x	y
0	10
1	9
2	8
3	7
10	0

Plotting these points, as in Fig. 16, A is the point that corresponds to the first pair of values, $x = 0, y = 10$. Similarly, B corresponds to $x = 1$, $y = 9$; C to $x = 2$, $y = 8$; D to $x = 3, y = 7$; and E to $x = 10$, $y = 0$.

Obviously by choosing values of x that differ from each other by very small fractions, the corresponding values of y will differ from each other also by very small fractions and we shall thus have a succession of points lying as close together as we choose. Apparently A, B, C, D, and E all lie in a *straight* line of indefinite length, and we infer * therefore that a *straight line of indefinite length,* of which AE is a part, is a representation of the equation $x + y = 10$. Such a figure is called a *graph.* We shall *assume,* therefore, that the *graph of a linear equation in two letters is a straight line.*

Moreover, since a straight line is determined by any two of its points, it will be sufficient in constructing the graph of a linear equation in two letters to plot any two points whose coordinates satisfy the given equation and then draw through these two points a straight line. The two points most conveniently plotted are generally the two in which the line cuts the axes. The coordinates

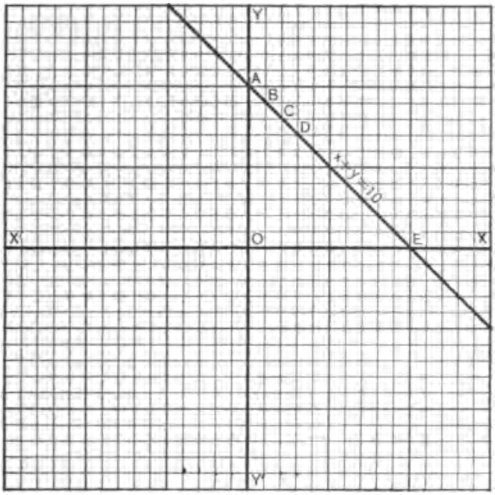

FIG. 16

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* It can be proved that the graph of any linear equation in two variables is a straight line, but such proof is beyond the scope of this book.

of these two points are found by putting, first x and then y equal to zero in the equation. Thus in the equation $x + y = 10$, putting $x = 0$, gives $y = 10$; and putting $y = 0$, gives $x = 10$. Therefore, the straight line cuts the x-axis in the point (10, 0) and the y-axis in the point (0,10). and these two points are sufficient to determine the graph.

EXERCISE*

(Use cross-section paper)

Construct the graph of each of the following:

1. $x + y = 5$.
2. $x - y = 3$.
3. $2x + 3y = 12$.
4. $3x - 2y = 0$.
5. $4x - 3y = 12$.
6. $3x - y = 0$.
7. $x = 5$.
8. $y = -2$.
9. $x = 0$.
10. $y = 0$.

To be answered orally:

11. Does the point (2, 3) lie on the line whose equation $x + y = 5$?

12. Give coordinates of other points that lie on the line whose equation is $x + y = 5$.

13. Does the point (5, 2) lie on the line whose equation is $x - y = 3$?

14. Give coordinates of other points that lie on the line whose equation is $x - y = 3$.

15. How do you determine from the coordinates of a point whether the point lies on a line whose equation is given or not?

16. Does the line whose equation is $3x - 2y = 0$ pass through the origin?

17. How do you tell from the form of an equation whether the graph will pass through the origin or not?

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* Tabulate the values of x and y as in § 62 in all cases, always putting x-values to the left of the y-values.

18. Where does the line whose equation is $x = a$ lie?

19. Where does the line whose equation is $y = -c$ lie?

20. Where does the line whose equation is $y = a$ lie?

63. The Graphical Solution of Two Linear Equations in Two Letters or Variables.

Let the two given equations be $x + y = 5$ and $x - y = 1$.

(1) The graph of each of these equations, constructed as in § 62, is shown in Fig. 17.

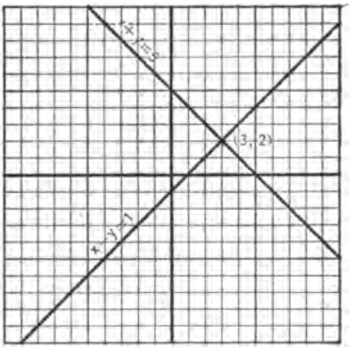

FIG. 17

(2) It will be observed that the two lines intersect at the point (3, 2).

(3) Since the point (3, 2) lies on both lines its coordinates must satisfy both equations. By actual trial this is found to be true; for $3 + 2 = 5$ and $3 - 2 = 1$. Therefore, the equations $x + y = 5$ and $x - y = 1$ have a common solution in $x = 3$ and $y = 2$.

Moreover, since two straight lines can intersect in only one point the two given equations can have but one common solution.

(4) A solution of equations obtained in this manner is called a graphical solution.

From the foregoing it will be seen that a graphical solution of two linear equations in two variables consists in (1) constructing the graphs of the two equations; (2) determining the coordinates of the point of intersection. The abscissa of this point of intersection is the value of x and the ordinate the value of y that satisfies both equations.

EXERCISE

Solve the following pairs of linear equations graphically:

1. $2x - y = 10.$
   $x - 2y = 2.$
2. $x + y = 7.$
   $-x + 2y = 5.$
3. $x - y = 9.$
   $x + 2y = 12.$
4. $2x - y = 1.$
   $x + y = 2.$
5. $5x - y = 19.$
   $x + 5y = -25.$
6. $2x - 7y = 9.$
   $5x + 3y = 2.$
7. $3x - 2y = 1.$
   $x + y = 1.$
8. $3x - 2y = 4.$
   $5x - 4y = 7.$
9. $2x + y = 0.$
   $4x + 6y = -3.$
10. $x - y = 2.$
    $x + y = -8.$

64. Simultaneous Linear Systems. It has been shown in § 53 that a linear equation in one unknown has but one value of the unknown number which will satisfy the equation. Such an unknown number is called a *constant.*

It has been shown in § 62 that a linear equation in two letters has an indefinite number of pairs of values of the unknown numbers which will satisfy the equation. Such unknown numbers are called *variables.*

Two or more equations with two or more unknown numbers constitute a *system* of equations. A system of equations is called *simultaneous* if the same values of the unknown numbers satisfy every equation of the system.

The solution of a set of two simultaneous linear equations by the graphical method has been given in § 63. We shall now proceed to the more general method of solving such equations known as the solution by elimination.

65. Elimination is a process used in getting rid of all the unknown numbers except one in a set of simultaneous equations. The general solution of simultaneous equations depends upon the elimination of unknown numbers, as illustrated in the following examples:

Illustrative Solution of Simultaneous Linear Equations.

Solve, (1) $x + 2y = 5.$ (2) $x - y = 2.$

Solution.

(1) $x + 2y = 5$, equation (1) as given.

(3) $2x - 2y = 4$, equation (2) multiplied by 2. Axiom 3: Equals multiplied by . . .).

(4) $3x = 9$, adding equations (1) and (3) together. (Axiom 1: Equals added to . . .).

(5) $x = 3$, dividing both sides by 3. (Axiom 4: Equals divided by equals . . .).

(6) $3y = 3$, subtracting equation (2) from equation (1). (Axiom 2: Equals subtracted from . . .).

(7) $y = 1$, dividing both sides by 3. (Axiom 4: as above).

Therefore, $x = 3$, and $y = 1$, is the required solution.

Check. $3 + 2 \cdot 1 = 5$, putting $x = 3$ and $y = 1$, in equation (1).
$3 + 2 = 5$, or $5 = 5$.

And $3 - 1 = 2$, putting $x = 3$ and $y = 1$ in equation (2).
$2 = 2$.

A study of the process of elimination leads to the following rule: *If the coefficients of either of the unknown numbers are equal, that number may be eliminated at once by addition or subtraction; and the resulting equation, which contains but one unknown number, may be solved in the manner of any linear equation containing but one unknown number. If the coefficients of either of the unknown numbers are unequal, either or both of the equations may be multiplied by such number or numbers as will make these coefficients equal. The equations thus obtained may be treated as in the first instance and a solution found.*

EXERCISE

66. Solve each of the following examples according to § 65. Check answers.

1. $2x - y = 10.$
$x - 2y = 2.$

2. $x + y = 7.$
$-x + 3y = 5.$

3. $\frac{x}{2} + \frac{y}{3} = 4.$
$\frac{x}{2} - \frac{y}{3} = 2.$

4. $7y - 4x = 59.$
$y + x = -1.$

5. $10y + 3x = 16.$
$y + 4x = -7.$

6. $.4v + .9a = 3.35.$
$v - 2a = 2.$

7. $s - v = 9.$
$s + 2v = 12.$

8. $4u - 3t = 34.$
$2u + t = 22.$

9. $2m - n = 1.$
$m + n = 2.$

10. $\frac{v}{2} + \frac{t}{3} = 6.$
$v + \frac{t}{3} = 9.$

11. $5r - s = 19.$
$r + 5s = -25.$

12. $1.6s - 2.6t = 10.$
$.04s + .75t = 10.03.$

In the following problems represent each of the unknown numbers by a different letter and solve the resulting simultaneous equations:

13. The sum of two numbers is 24 and their difference is 10. Find the numbers. (Suggestion. Let n' and n'' represent the numbers. Then $n' + n'' = 24$, etc.,

14. The sum of two numbers is s and their difference is d. Find the numbers.

15. A farmer bought 100 acres of land for $4500. He paid $30 an acre for part of it, and $60 an acre for the remainder. How many acres of land did he buy at each price?

16. Tickets to a school entertainment were sold to adults at 50 cents each and to children at 25 cents each. The admission receipts were $110 and the number of tickets sold was 250. How many tickets of each kind were sold?

17. A lot whose length exceeds its width by 40 feet has a perimeter of 240 feet. What are its dimensions? (Compare with examples in § 57.)

18. The 600 employees in a factory receive $4800 in wages every Saturday night. The men's wages average $2 per day and the wages of all other employees average $1 per day. How many men are employed?

19. At a certain primary election there were 324 votes cast and the nominee had a majority of 34. If there were but two candidates, how many votes did each receive?

20. Fifty laborers were engaged to work upon a bridge; some of them were to receive $1.80 per day, others were to receive $3 per day. On completing the work it was found that all together received $96. How many laborers were employed at each rate?

21. Two boys start for school at the same time; one lives half a mile east of the schoolhouse and the other lives 112 rods west of it. If the first walks at the rate of 32 rods per minute and the second at the rate of 22 rods per minute, in how many minutes will they be at equal distances from the schoolhouse?

22. A merchant has two kinds of tea that cost 30 cents and 40 cents respectively per pound, and desires to make up a mixture containing 50 pounds that will cost him 32 cents per pound. How many pounds of each kind must he use?

23. Commercial alcohol is 95% pure. How many quarts of water must be added to 20 quarts of such alcohol in order that it may be reduced to 80% pure?

24. If 100 pounds of sea-water contain 3 pounds of salt, how much fresh water must be added to 200 pounds of sea-water in order that the new mixture may contain only 1 pound of salt to 100 pounds of water?

25. The perimeter of a football field is 920 feet and the length is 20 feet less than twice the width. What are its dimensions?

26. The distance from Chicago to St. Louis is 300 miles. Two automobiles start out at the same time, one from each city, and run at the rate of 15 and 20 miles per hour respectively, until they meet. How many miles will each have gone when they meet?

27. The total land area of the United States and its possessions is approximately 3,750,000 square miles; the area of the United States is 1,500,000 square miles more than double that of its outlying possessions. Find the area of the United States.

28. In 100 pounds of milk of a certain grade the liquid part weighs 7 times as much as the solids; the solids other than butter-fat weigh 3 times as much as the butter-fat. How many pounds of butter-fat are contained in the 100 pounds of milk?

29. In the reduction of iron ore, a mixture of the ore with limestone and coke is put into the smelter. If on a certain day 20,000 tons of the mixture were put into the furnace, and if the quantity of coke used was three times the quantity of limestone, and the quantity of limestone was six times the quantity of ore, how many tons of each were used?

30. For the year 1911 the average earnings per mile of a certain system of railroad were $4500. The freight earnings were $500 per mile better than the passenger earnings and the latter were double the earnings from all other sources such as rentals, express, mail, etc. What were the freight and passenger earnings per mile?

CHAPTER V

MULTIPLICATION AND DIVISION—LINEAR SYSTEMS IN TWO UNKNOWNS

(a) Multiplication

67. Principles. We learned in arithmetic that $4 \times 5 = 5 \times 4$.

We shall now assume that

I $$\mathbf{a \times b = b \times a},$$

where a and b represent any two numbers whatever. This means that *the product of two numbers is the same in whatever order they may be taken.*

We also learned in arithmetic that $3 \times (2 \times 5) = (3 \times 2) \times 5$; in other words, that $3 \times 10 = 6 \times 5$. We shall now assume that

II $$\mathbf{a \times (b \times c) = b \times (c \times a)}.$$

This means that *the product of several numbers is the same in whatever way they may be grouped.*

Again, we learned in arithmetic that $3 \times (4 + 5) = 3 \times 4 + 3 \times 5$; in other words, that $3 \times 9 = 12 + 15$. And we shall now assume that

III $$\mathbf{a \times (x + y) = ax + ay},$$

where a, x, and y are any numbers whatever. This means that *the product of the sum of two or more numbers by* *another number is the same as the sum of the products obtained by multiplying each of these numbers separately by the other number.*

From arithmetic we know that $0 \times 2 = 2 \times 0 = 0$.

In general we shall assume that

$$\text{IV} \qquad \mathbf{0 \times a = a \times 0 = 0.}$$

68. Law of Exponents in Multiplication. From the definition of integral exponent, § 14, we have,

$$3^2 = 3 \times 3;$$

and $$3^4 = 3 \times 3 \times 3 \times 3.$$

Hence, $$\begin{aligned} 3^2 \times 3^4 &= (3 \times 3) \times (3 \times 3 \times 3 \times 3) \\ &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \\ &= 3^6. \end{aligned}$$

In general,

$$\begin{aligned} a^2 \times a^3 &= (a \times a) \times (a \times a \times a) \\ &= a \times a \times a \times a \times a \\ &= a^{2+3} = a^5. \end{aligned}$$

Therefore, when p and q are integers,

$$\text{V} \qquad \mathbf{a^p \cdot a^q = a^{p+q}.}$$

This means that *the exponent of any letter in the product equals the sum of the exponents of that letter in the terms to be multiplied together.*

It is very important to observe here that when letters are used to represent numbers in algebra, different letters represent different numbers; and hence, $a^2 \cdot b^3$ does not equal a^5 nor b^5, any more than that $2^2 \cdot 7^3 = 2^5$ or 7^5.

69. Law of Signs for Multiplication as stated in § 30, is as follows: *Like signs produce positive results; unlike* *signs produce negative results,* whether the coefficients be literal or numerical.

70. Multiplication of Monomials.

Example. Multiply $4\,a^3c^2$ by $-3\,abc$.

Solution. $4\,a^3\,c^2 \times -3\,abc = -12\,a^4bc^3$.

Explanation. The sign of the product is $-$ since the two terms to be multiplied together have unlike signs. The exponent of a in the product is 4, since that is the sum of the exponents of a^3 and a. The exponent of b in the product is the same as in the given term, since it does not occur in the other term. The exponent of c in the product is 3, since 3 is the sum of the exponents of c in the given terms.

Check.

Let $a = b = c = 1$.
Then $4\,a^3c = 4 \times 1 \times 1 = 4$,
and $-3\,abc = -3 \times 1 \times 1 \times 1 = -3$.
$4 \times (-3) = -12 =$ product of numerical values.
But $-12\,a^4bc^2 = -12 \times 1 \times 1 \times 1 = -12 =$ numerical value of product.

Therefore, since the product of the numerical values is the same as the numerical value of the product, we conclude that the result is correct.

In like manner we may show that,

$(-2\,ab^2c) \times (-3\,a^2bc^2) = 6\,a^3b^3c^3$.

Also, that $(-2\,ab) \times (-3\,ab) \times (-4\,ab) = -24\,a^3b^3$.

(First multiply $-2\,ab$ by $-3\,ab$ and then that product by $-4\,ab$.)

In like manner,

$(-6\,ac) \times (-2\,ac) \times (-ac) \times (-4\,ac) = 48\,a^4c^4$.

Observe that an even number of $-$ factors produces a $+$ product; an odd number of $-$ factors produces a $-$ product. For, an even number of $-$ factors may be multiplied together two at a time and all the factors used; these $+$ products may then be multiplied together, giving a final $+$ product. On the other hand, an odd number of − factors cannot all be used in multiplying them two at a time as there would remain one − factor unused. This remaining − factor multiplied into the other products, which would be positive as before, will give a final − product.

In the multiplication of monomials it will be seen from the foregoing solution that:

(1) *The coefficient of the product is obtained by multiplying together the coefficients of the given monomials, due attention being given to the signs.*

(2) *The exponent of any letter in the product is the sum of the exponents of that same letter in the given monomials.*

EXERCISE

Perform the multiplications indicated and check results by substituting numerical values for letters used.

1. $-5a^2b \times 3b^2c$.
2. $10m^2n^2 \times -2m^5$.
3. $-mnpq \times -3p^5qRs$.
4. $4gt \times -\frac{3}{4}gt^2$.
5. $7ac^2 \times 3a^2b \times -2ab^2c^5$.
6. $-\frac{3}{5}x^3y^2 \times \frac{2}{3}xy^3 \times -5xy$.
7. $-3cd \times -4de \times -\frac{1}{2}ce$.
8. $2 \cdot 5m^3n^3 \times 3 \cdot 2m^3n \times -1 \cdot 5mn^2$.
9. $-2ab \times -abc \times -10c \times -5ab$.
10. $3(a-b) \times -2(a-b)$.
11. $-7(x^2-y^2) \times -4(x^2-y^2)$.
12. $-3(m+n) \times -(m+n)$.
13. $c(x-y) \times d(x-y) \times -e(x-y)$.
14. $10\left(\frac{xy}{2}\right) \times -2\left(\frac{xy}{2}\right)$.
15. $a^m \times a^n$.
16. $a^x \times a^n \times a^c$.
17. $-2a^m \times 3a^n \times a^p$.
18. $10a^m \times -4a^m$.
19. $-a^m \times a^{2m} \times a^3$.
20. $-5y^m \times 4y^{-2m} \times y^{-m}$.

71. Multiplication of Polynomial by Monomial.

Example. Multiply $2x^2 - 3y$ by $4xy$.

Solution.
$$\begin{aligned} 4xy \times (2x^2 - 3y) &= 4xy \times 2x^2 + 4xy \times (-3y) \\ &= 8x^3y - 12xy^2. \end{aligned}$$

The work for this problem is often arranged as below:

$2x^2 - 3y$	number to be multiplied
$4xy$	number by which we multiply
$8x^3y - 12xy^2$	product

Check. Let $x = y = 1$.

Then
$$\begin{aligned} 2x^2 - 3y &= 2 - 3 &&= -1 \\ \underline{4xy} &\underline{= 4 \times 1 \times 1} &&\underline{= 4} \\ & &&-4 = \text{product of numerical values.} \end{aligned}$$

But $8x^2 - 12xy^2 = 8 - 12 = -4$ = numercial value of product.

Therefore, since the product of the numerical values of the numbers to be multiplied is the same as the numerical value of the product, we conclude that the result is correct.

The following rule for the multiplication of a polynomial by a monomial is based upon the foregoing solution:

(1) *Arrange the terms of the polynomial in any convenient order.*

(2) *Multiply each term of the polynomial by the monomial as in* § 70.

(3) *Collect these products into a single expression.*

EXERCISE

Multiply and check each of the following:

1. $-7ab^2(3ac - 2a^2bc + abc)$.
2. $-xy(2x^2 - 4xy + 2y^2)$.
3. $10m^2n^2(2m^2 + 4mn + 2n^2)$.
4. $25(a^2 + 2ab + b^2)$.
5. $12a^2(4a^2 + 12ab + 9b^2)$.
6. $-3xy(mn + ab - pq)$.

7. $\frac{3}{4}\left(\frac{a^2}{4} - ab + \frac{b^2}{4}\right)$. 8. $-3x(ax - bx^2 + 3cx^3)$.

9. $-7bc(ay^2 + by - 3cy)$.

10. $\frac{7}{3}(ax^3)(-9a - 12x - 42ax)$.

72. Multiplication of Polynomials.

Example. Multiply $a^2 + ab + b^2$ by $a + b$.

$$\begin{array}{rl} \textit{Solution.}\quad a^2 + ab + b^2 & \text{first number.} \\ \underline{\qquad\qquad a + b} & \text{second number.} \\ a^3 + a^2b + ab^2 \qquad\quad & \text{product of } (a) \text{ and first number.} \\ \underline{a^2b + ab^2 + b^3} & \text{product of } (b) \text{ and first number.} \\ a^3 + 2a^2b + 2ab^2 + b^3 & \text{adding similar terms.} \end{array}$$

Check. Let $a = b = 1$.

$$\begin{array}{lrcl} \text{Then} & a^2 + ab + b^2 = 1 + 1 + 1 & = & 3 \\ \text{And} & a + b = 1 + 1 & = & 2 \\ \hline & & & 6 = \text{product of the numerical values.} \end{array}$$

But $a^3 + 2a^2b + 2ab^2 + b^3 = 1 + 2 + 2 + 1 = 6 =$ numerical value of product.

Therefore, since the product of the numerical values is the same as the numerical value of the product, we conclude that the result is correct.

Observations upon the foregoing solution lead to the following rule for the multiplication of polynomials:

(1) *Arrange the terms of both expressions according to the powers of some letter contained in those terms.*

(2) *Multiply all the terms of one expression in succession by each of the terms of the other and add resulting products.*

EXERCISE

Perform the multiplications indicated and check results.

1. $\begin{array}{r} x^3 + x^2y + xy^2 + y^3 \\ \underline{x - y} \end{array}$ 2. $\begin{array}{r} 3a^3 - ab^2 + b^3 \\ \underline{2ab + b^3} \end{array}$

3. $$\begin{array}{r} 1 - 2x + 3x^2 \\ 1 - x \\ \hline \end{array}$$

4. $$\begin{array}{r} a^4 + a^3b + a^2b^2 + ab^3 + b^4 \\ a - b \\ \hline \end{array}$$

5. $$\begin{array}{r} 2a - a^2b + 4a^3b^2 - a^4 \\ 1 - b - a^2 \\ \hline \end{array}$$

6. $$\begin{array}{r} m^4 + 4m^3n + 6n^2m^2 + 4mn^3 + n^4 \\ m^2 + 2mn + n^2 \\ \hline \end{array}$$

7. $$\begin{array}{r} x^2y + x^3 + xy^2 + y^3 \\ xy + x^2 - y^2 \\ \hline \end{array}$$

First arrange terms as below.

$$\begin{array}{r} x^3 + x^2y + xy^2 + y^3 \\ x^2 + xy - y^2 \\ \hline \end{array}$$

8. $(v^2t + vt^2 + v^3 - 3)(vt^2 - 3t^2 + 4)$.

9. $(4x^4 - 12x^2y^2 + 9y^4)(x^2 - 2xy + y^2)$.

10. $(at + \frac{1}{2}at^2 + \frac{1}{4}at^3 + \frac{1}{8}at^4)(2at - 4at^2)$.

11. $\left(\frac{b}{2} + \frac{c}{3} - \frac{d}{4}\right)\left(\frac{b}{2} + \frac{c}{3} + \frac{d}{4}\right)$.

12. $(a^{2n} + 2a^nb^n + b^{2n})(a^{2n} - 2a^nb^n + b^n)$.

13. $(3^x + 2^y)(3^x - 2^y)$.

14. $(5^c + a^e)(5^c - 2a^e)$.

15. $(a^2 - \frac{1}{2}a + \frac{1}{4})(a^2 + a - \frac{1}{4})$.

16. $(x^2 - xy + y^2)(x + y)$.

17. $(-cy + d)(ay - cy^2)$.

18. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

19. $(a^x - b^y)(a^x + b^y)$.

20. $(a^x + b^y + c^p)(a^{2x} + b^{2y} - c^{2p})$.

(b) Division

73. Principles. We learned in arithmetic that $3 \div 4$ may be expressed as a fraction $\frac{3}{4}$. In general, we shall assume that $a \div b$ may be written as a fraction $\frac{a}{b}$. Also that $a^5 \div a^3 = \frac{a^5}{a^3} = \frac{\cancel{a} \cdot \cancel{a} \cdot \cancel{a} \cdot a \cdot a}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a \cdot a = a^2$.

VI Hence, $\mathbf{a^p \div a^q = a^{p-q}}$.

For the present we shall assume that p is always greater than q.

This means that *the exponent of any letter in the quotient is found by subtracting the exponent of that letter in the divisor from the exponent in the dividend.*

Again, we observed in § 67 that

$$0 \times a = a \times 0 = 0.$$

Hence it follows that

VII $\mathbf{0 \div a = 0}$, if $\mathbf{a \neq 0}$.

This means that *zero divided by any number other than zero is zero.*

But $a \div 0$ and $0 \div 0$, that is, division by 0 is excluded in this discussion.

As noted in § 30, the law of signs is the same for division as for multiplication: *Like signs produce positive and unlike signs produce negative results.*

74. Division of Monomials.

Example. Divide $20\,a^3b^4$ by $4\,a^2b$.

Solution. $20\,a^3b^4 \div 4\,a^2b = 5\,ab^3$.

For,

$$20 \div 4 = 5,$$
$$a^3 \div a^2 = a \ (\S\ 73),$$
$$b^4 \div b = b^3 \ (\S\ 73),$$

Check. $20\,a^3b^4 = 20$, if $a = b = 1$.

$$4\,a^2b = 4,$$

and $20 \div 4 = 5$, quotient of numerical values.

But $5\,ab^3 = 5$, if $a = b = 1$.

Therefore, since the quotient of the numerical values is the same as the numerical value of the quotient, we conclude that the result is correct.

As observed in the foregoing solution the quotient of one monomial divided by another may be obtained as follows:

(1) *The coefficient of the quotient is found by dividing the coefficient of the dividend by the coefficient of the divisor.*

(2) *The exponent of any letter in the quotient is found by subtracting the exponent of that letter in the divisor from the exponent in the dividend.*

EXERCISE

Perform the operations indicated and check as above.

1. $25\,a^4b^3c^2 \div 5\,a^2bc$.
2. $-64\,m^5n^3 \div 16\,mn$.
3. $-120\,x^7y^5z^4 \div -15\,x^2y^2z^4$.
4. $a^4b^5 \div -a^3b^2$.

(Suggestion: $z^4 \div z^4 = z^0 = 1$, since any number is contained in itself once.)

5. $0 \div 5\,m^2n$.
6. $(-3\,a^2b)^4 \div 9\,a\,b$.
7. $(-3)^6 \div (-9)^2$.
8. $(x - y)^5 \div -(x - y)$.

9. $\dfrac{(a^2 - b^2)^3}{a^2 - b^2}$.

10. $a^{4m} \div a^{3m}$.

11. $a^m \div a$.

12. $-10\,x^a y^c \div 2\,x^b y^{2c}$.

13. $3\,a^m c^n \div -ac$.

14. $-9\,x^{2a} y^c \div -3\,x^2 y$.

15. $m^4(x^2 - y^2)^3 \div -m^2(x^2 - y^2)$.

16. $-10\,a^2b^3 \times 4\,a^5b^4c^2 \div -8\,a^2bc^2$.

17. $m^2n^3 \times -16\,m^4n^2 \div 4\,m^2n^2$.

18. $12\,a^m c^n \times 4\,a^{2m}c^{3n} \div 16\,a^2c$.

19. $a^x(x^m - y^m)^3 \times a^2b^2\,(x^m - y^m) \div a^{ax}(x^m - y^m)$.

20. $x^m y^n z^2 \times x^{3m} y^n z^4 \div x^m y^{2n} z^3$.

75. Division of Polynomial by Monomial.

Example. $(12\,x^4y^3 - 20\,x^5y^3 + 8\,x^4y^4) \div 4\,x^2y^2$.

Solution. $12\,x^4y^3 \div 4\,x^2y^2 = 3\,x^2y$. (§ 74.)
$-20\,x^5y^3 \div 4\,x^2y^2 = -5\,x^3y$. (§ 74.)
$8\,x^4y^4 \div 4\,x^2y^2 = 2\,x^2y^2$. (§ 74.)

Adding terms we have $3\,x^2y - 5\,x^3y + 2\,x^2y^2$

as the quotient of $(12\,x^4y^3 - 20\,x^5y^3 + 8\,x^4y^4) \div 4\,x^2y^2$.

Check. Let $x = y = 1$.

Then $12\,x^4y^3 - 20\,x^5y^3 + 8\,x^4y^4 = 12 - 20 + 8 = 0$,

and $4\,x^4y^4 = 4 \times 1 \times 1 = 4$.
$0 \div 4 = 0$, (VII).

But $3\,x^2y - 5\,x^3y + 2\,x^2y^2 = 0$, if $x = y = 1$.

Therefore, since the quotient of the numerical values is the same as the numerical value of the quotient, we conclude that the result is correct.

From the foregoing solution we have the following rule for the division of a polynomial by a monomial:

(1) *Divide each term of the polynomial in succession by the monomial as in § 74, giving due attention to signs.*

(2) *Add the resulting quotients by bringing them down in succession, each with its own sign.*

EXERCISE

Perform operations as indicated, checking each result.

1. $(x^3 - 3x^2y + 3xy^2) \div x$.
2. $(4a^3b^2c^2 - 20a^4b^3c + 16a^3b^3c^3) \div 4a^3bc$.
3. $8(a^3 - b^3)^3 - 6(x^2 + y^2) - 12(m^4 - n^4) \div -2$.
4. $(x^m - y^n)^4 + (x^m - y^n)^2 - (x^m - y^n)^3 \div (x^m - y^n)^2$.
5. $(-a^mb^nc^n + 2a^{2m}b^{3n}c^{4n} - 4abc) \div abc$.
6. $\dfrac{3m^2n^2 + 4mnc + 8m^3n^3c^3}{2mn}$.
7. $\dfrac{vt + at^2 - 3at^3}{t}$.
8. $\dfrac{nx^c + bx^n + dx^m}{x^c}$.
9. $\dfrac{a^{m-n} + 5a^2 - 3a^nb - a^{2m+n}}{a^{m+1}}$.
10. $\dfrac{-10x^3y^m - x^ay^2 + 7x^c - 2y^8}{x^2y^3}$.

76. Division of Polynomial by Polynomial.

Example 1. Divide $a^3 + 3ab^2 + b^3 + 3a^2b$ by $a^2 + 2ab + b^2$.

Arranging terms according to decreasing powers of a.

$a^3 + 3a^2b + 3ab^2 + b^3 \,|\, \underline{a^2 + 2ab + b^2}$ dividend and divisor.

1st term 2d term.

$|\underline{a + b}$ quotient.

$\underline{a^3 + 2a^2b + ab^2}$ product of divisor and first term of quotient.

$a^2b + 2ab^2 + b^3$ remainder with b^3, last term of dividend brought down.

$\underline{a^2b + 2ab^2 + b^3}$ product of divisor and second term of quotient.

Since there is no remainder and since all the terms of the dividend have now been used, the division is complete.

Explanation. Having arranged terms according to decreasing powers of a, the first term of the quotient, a, was found by dividing the first term of the dividend, a^3, by the first term of the divisor a^2. Thus, $a^3 \div a^2 = a$. The product of the divisor $a^2 + 2\,ab + b^2$, and a, the first term of the quotient, was found to be $a^3 + 2\,a^2b + ab^2$, which was written underneath the dividend so that similar terms should stand in the same column and subtracted. b^3, the last term in the dividend, was then brought down along with the remainder, forming the expression, $a^2b + 2\,ab^2 + b^3$. The first term of this expression, a^2b, was then divided by a^2, the first term of the divisor, giving b, which was written down as the second term of the quotient. The product of the divisor $a^2 + 2\,ab + b^2$ and b, the second term of the quotient, was found to be $a^2b + 2\,ab^2 + b^3$, which was written underneath the corresponding similar terms of the remainder and subtracted. There being no remainder this time and all the terms of the dividend having been brought down and used, the division is said to be complete and the quotient is $a + b$. Moreover, since there is no final remainder the division is *exact*.

Check. Let $a = b = 1$.

Then $a^3 + 3\,a^2b + 3\,ab^2 + b^3 = 1 + 3 + 3 + 1 = 8$,

and $a^2 + 2\,ab + b^2 = 1 + 2 + 1 = 4$,

$$8 \div 4 = 2,$$

But $a + b = 2$, if $a = b = 1$.

Therefore, since the quotient of the numerical values is the same as the numerical value of the quotient, we conclude that the result is correct.

Another method of checking in division is to multiply divisor and quotient together, adding to this product the remainder, if any. This result should be the same as the dividend if the work is correct.

Example 2. The division of $x^3 + 3x^2y + y^3$ by $x + y$ shows a final remainder; that is, the division is not exact. The work is given below.

Arranging terms according to decreasing powers of x and observing that the first power term in x is missing, and may therefore be considered as a term with zero coefficient, we have:

$x^3 + 3x^2y + 0xy^2 + y^3$	$x + y$ dividend and divisor as arranged.
	1st term 2d term 3d term
	$x^2 + 2xy - 2y^2$ quotient.
$x^3 + x^2y$	product of $(x + y)$ and (x^2).
$2x^2y + 0xy^2 + y^3$	1st remainder with dividend terms brought down.
$2x^2y + 2xy^2$	product of $(x + y)$ and $(2xy)$.
$-2xy^2 + y^3$	2d remainder.
$-2xy^2 - 2y^3$	product of $(x + y)$ and $(-2y^2)$.
$+3y^3$	final remainder, since, not containing x, it can no longer be divided by x, the first term of the divisor.

The quotient then of $(x^3 + 3x^2y + y^3) \div (x + y)$ is

$$x^2 + 2xy - 2y^2 + \frac{3y^3}{x + y}.$$

Observe that the last term of this result is the remainder written over the divisor as a fraction and is preceded by the + sign of the remainder.

The following rule for division of polynomials is based upon observations of the foregoing examples:

(1) *Arrange terms of dividend and divisor from left to right according to the decreasing powers of some common letter.*

(2) *Divide the first term of the dividend by the first term of the divisor to get the first term of the quotient.*

(3) *Multiply each term of the divisor by the first term of the quotient; write the terms of the product under the similar*

terms of the dividend and subtract. If in this product there should be a term that is not similar to any of the dividend terms, place it at the right of other terms in the product and, when subtracting, bring down this term with its sign changed.

(4) *If there be a remainder, consider it together with the next term of the dividend as a new dividend and proceed as before, keeping the terms arranged in the new dividend as they were in the first dividend.*

(5) *If the division is not exact, write the final remainder over the divisor in the form of a fraction at the right of the quotient, prefixing to the fraction the sign of the remainder.*

EXERCISE

Perform the following divisions and check results:

1. $(m^2 - 8m + 15) \div (m - 5)$.
2. $(a^2 - a - 12) \div (a - 4)$.
3. $(64a^4b^4 - 16a^2b^2 + 1) \div (8a^2b^2 - 1)$.
4. $(32x^2 + 28xy - 15y^2) \div (4x + 5y)$.
5. $(y^3 - 8y^2 - 5y + 84) \div (y - 7)$.
6. $(v^3 - 5v^2a + 9va^2 - 9a^3) \div (v^2 - 2va + 3a^2)$.
7. $(a^5 + 7a^3 + 14a - 15 - 11a^2 - 2a^4) \div (a^2 - a + 3)$.
8. $(-8 - 8x^2 + 4x - 2x^3 + 4x^4) \div (2 + 3x^2 - x)$.
9. $(56x^4 + 119x^2y^2 + 45y^4 - 47xy^3 - 53x^3y)$
$\div (-4xy + 7x^2 - 9y^2)$.
10. $(1 - 18m^2 + 81m^4) \div (9m^2 - 6m + 1)$.
11. $(a^{3m} - 3a^{2m}b^n + 3a^mb^{2n} - b^{3n}) \div (a^m - b^n)$.
12. $(m^{a+3} + 3m^{a+2} + 3m^{a+1} + m^a + 1) \div (m^{a+1} + 1)$.
13. $(a^{3m} - b^{3n}) \div (a^m - b^n)$.
14. $\left(\frac{a^3}{8} + \frac{a^2b}{4} + \frac{ab^2}{6} + \frac{b^3}{27}\right) \div \left(\frac{a}{2} + \frac{b}{3}\right)$.

15. $(8a^6 - 27v^3) \div (2a^2 - 3v)$.

16. $(m^4 - m^2n^2 + n^4) \div (m^2 + mn + n^2)$.

17. $(x^3 + 3x^2y - 3xy^2 + y^3) \div (x + y)$.

18. $(x^{2a} + 5x^a + 6) \div (x^a + 2)$.

19. $(a^{6p} - 7a^{3p} + 12) \div (a^{3p} - 4)$.

20. $(x^5 - y^5) \div (x - y)$.

(c) Simultaneous Linear Equations

77. The Graphical Solution of simultaneous linear equations has been considered in § 63; and in § 65 another method of solution, known as *elimination by addition and subtraction,* has been studied. We shall next consider the solution of these equations by the methods known as *elimination by substitution, and elimination by comparison.*

Example 1. Let (1) $x - y = 10$,
and (2) $2x + 3y = 30$ be two simultaneous equations.

Solve for x and y, eliminating by substitution.

Solution.

(1) $x = 10 + y$, adding y to both sides of equation (1).

(3) $2(10 + y) + 3y = 30$ by substituting in equation (2) the above value of x from equation (1).

(4) $20 + 2y + 3y = 30$, removing parentheses in equation (3).

(5) $2y + 3y = 30 - 20$, subtracting 20 from both sides of equation (4).

(6) $5y = 10$, collecting terms in equation (5).

(7) $y = 2$, dividing both sides by 5, the coefficient of y.

(8) $x - 2 = 10$, substituting the above value of y in equation (1).

$x = 10 + 2$, adding 2 to both sides of equation (8).

$x = 12$, by combining similar terms.

Therefore, $x = 12$ and $y = 2$, are the required values.

Check. $12 - 2 = 10$ or $10 = 10$, equation (1).
$2 \cdot 12 + 3 \cdot 2 = 30$, or $24 + 6 = 30$, or $30 = 30$, equation (2).

Example 2. Let (1) $x - y = 10$,
and (2) $2x + 3y = 30$, be the same two simultaneous equations as in Example 1, to be solved for x and y, eliminating by comparison.

Solution.

(1) $x = 10 + y$, adding y to both sides of equation (1).

(2) $2x = 30 - 3y$, subtracting $3y$ from both sides of equation (2).

And $x = \frac{30 - 3y}{2}$, dividing by 2, the coefficient of x.

Hence,

(3) $10 + y = \frac{30 - 3y}{2}$, since both these expressions represent the value of x.

Solving equation (3) for y,

(4) $20 + 2y = 30 - 3y$, multiplying both sides of equation (3) by 2.

(5) $2y + 3y = 30 - 20$, adding $3y$ and subtracting 20 from both sides of equation (4).

(6) $5y = 10$, by combining similar terms in equation (5).

(7) $y = 2$, dividing both sides of equation (6) by 5, the coefficient of y.

(8) $x - 2 = 10$, by substituting in equation (1), the above value of y.

(9) $x = 10 + 2$, adding 2 to both sides of equation (8).

(10) $x = 12$, by combining similar terms in equation (9).

Therefore, $x = 12$ and $y = 2$ are the required values.

It will be seen that these results are the same as found by the preceding solution.

EXERCISE

Solve each of the following examples and problems by the method that seems most readily adapted. Check each solution.

1. $x + y = 8$,
$3x + 2y = 23$

2. $9s = \frac{y}{3} + \frac{23}{4}$,
$\frac{7y}{3} - \frac{9}{4} = \frac{-3s}{4}$.

3. $x - y = 4,$
$2x + 3y = 13.$

4. $\dfrac{.2t + .5}{1.5} = \dfrac{.49u - .7}{4.2},$
$\dfrac{.5u - .2}{1.6} = \dfrac{4t - 1.5t + 6.7.}{6.4}$

5. $12x - 10y = 14,$
$y - 2x = 3.$

6. $.08m + .9n = .048,$
$.3m - .35n = .478.$

7. $7x - 5y = 52,$
$4x + 10y = 94.$

8. $5r - \frac{1}{3}(3r - 2s + 5) = 11,$
$\frac{5}{6}(r - 4s) - \frac{4}{9}(r - s) = 16.$

9. $3x_1 - 18x_2 = 21,$
$12x_2 - x_1 = 0.$

10. $14m - 2n = 1,$
$n - 6m = 0.$

11. $2r_0 + 25r_1 = 15,$
$9r_0 - 30r_1 + 132 = 0.$

12. $\dfrac{3y - 20}{2} = \dfrac{2y + 5x}{3},$
$10 = y - x.$

13. $\frac{2}{3}(s' - 2s) + 18 = 0,$
$.5(4s' - s) = .5.$

14. $.4y + .9x = 5.7,$
$2y - x = 1.$

15. $12y = \dfrac{11z}{9} + 17,$
$\dfrac{2y}{3} - \dfrac{49}{12} = \dfrac{5z}{12}.$

16. $s - 5n = 4s - 6n,$
$s - (5n - s) + (s + 4) = 0.$

17. $\dfrac{5x' + 2y'}{2} = y' + 2,$
$x' - \dfrac{2y'}{3} = 0.$

18. $10x' + 3x'' = 84,$
$16x' - 3x'' = 72.$

PROBLEMS—ONE OR TWO UNKNOWN NUMBERS

78. **1.** A baseball team's schedule calls for 48 games, of which 36 have already been played. If 25 of the games played have been won, how many of the remaining games must be won in order that its per cent of games won in the whole series may be 75%?

2. A stock broker sold 9 shares of N. Y. Central stock and 7 shares of I. C. stock for \$1702; and at the same time another broker sold 6 shares of N. Y. Central and 5 shares of I. C. stock for \$1201. If the selling price per share for each kind of stock was the same in both instances, what was the price per share?

3. Weights of 12 and 18 pounds are fastened to the ends of a 12-foot teeter board.* Where must the support be placed in order that the board may balance?

* The principle involved in the teeter board and the ordinary scale beam is the same as that of a lever of the first class. (See Fig. 18). The ordinary lever is a bar, carrying weights w' and w''. In levers of the first class, w' and w'' are placed at the ends of the bar, and the support for the bar is placed between the weights. In the figure the support is attached at F, and is commonly called the fulcrum. In order that two unequal weights may balance, the lighter one must be placed at a greater distance from the fulcrum than the heavier. (See Fig. 19.) Designating those distances as d' and d'', the equation of the lever is

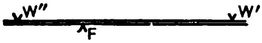

FIG. 18

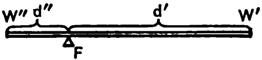

FIG. 19

$$w'd' = w''d''$$

or in words: *the products of the weights by their respective distances from the fulcrum are equal.*

It will be seen that this equation contains four quantities; any three of these being known the fourth may be found from the equation.

Thus, to find at what distance a weight of 10 lbs. must be placed to balance a weight of 25 lbs. placed at a distance of 8 ft.:

Let $w' = 10$, $w'' = 25$, and $d'' = 8$.

Then $10\,d' = 25 \cdot 8$, substituting in the equation of the lever.

$10\,d' = 200$, performing the operation indicated.

$d' = 20$, dividing both sides by 10.

Therefore, w', the 10-lb. weight, must be placed 20 feet from the fulcrum in order to balance the 25-lb. weight placed at a distance of 8 feet from the fulcrum.

4. An unknown weight is placed 3 inches from the point of support of a scale beam and is balanced by another unknown weight placed 9 inches from the point of support. If 2 pounds be added to the first weight, the second weight will have to be moved three inches farther out from the point of support in order to preserve the balance. Find the two weights.

5. Two unknown weights balance when placed 36″ and 54″ from the fulcrum of a lever; if their positions are reversed, 2 pounds must be added to the lesser weight to restore the balance. What are the weights?

6. A train leaves St. Paul 40 minutes late and runs to Sioux City at a speed $\frac{1}{5}$ greater than its schedule, arriving 16 minutes late. A second train follows, running under the same conditions except that it is unable to maintain the increased speed for the last 15 miles, which is covered at the usual speed, and arrives 22 minutes late. Find the usual speed and the distance between the two cities.

7. The circumference of the fore wheel of a wagon is 2 feet less than that of the rear wheel. If the fore wheel makes as many revolutions in going 100 yards as the rear wheel makes in going 120 yards, find the circumference of each wheel.

8. *AB* (Fig. 20) is a bar of iron 6 feet long called a crowbar. The end *A* is placed under a box weighing 1200 lbs. and the support is placed at *F*, 6 inches from *A*. How much loading (weight) must be applied at *B* to raise the box?

9. What loading would have to be applied at *B*, if *AF* were 3 inches instead of 6 inches?

10. Figure 21 illustrates the common suction pump used for lifting water out of wells and cisterns. *AB* is the handle and is

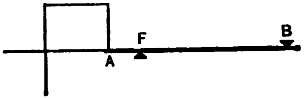

FIG. 20

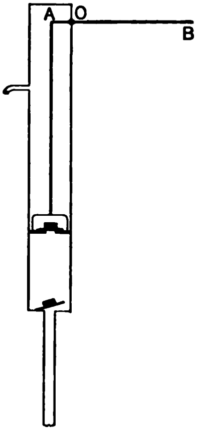

FIG. 21

supported at O by a pin. The piston attached to A raises the water. If $OA = 3$ inches and $OB = 3$ ft., what load applied at B will lift a load of 1000 lbs. on the piston?

11. In the preceding problem let OA be $1\frac{1}{2}$ inches. What load at B will balance a load of 1000 lbs. on the piston?

12. Suppose OA equals $1\frac{1}{2}$ inches and OB equals $1\frac{1}{2}$ feet. What load at B will balance a load of 1000 lbs. on the piston?

13. For what distance OA will 100 lbs. at B balance 1000 lbs. at A?

14. The Panama Canal has three systems of locks, viz., Gatun, Pedro Miguel, and Miraflores. The system at Gatun, consisting of three locks, effects a total lift of 85 feet. The middle and lower locks lift a vessel the same height while the upper locks lift it 3 feet less than each of the others. What is the lift of each lock?

15. The combined base areas of three 550-foot locks at Gatun is 163,000 sq. ft. If the two lower locks have equal bases and the base of the upper lock exceeds that of the lower ones by 2000 sq. ft., what are their respective base areas?

16. The number of gates required for the Gatun lock is 3 less than half the number required for the whole canal and 8 more than the number at Pedro Miguel. The number of gates at Miraflores exceeds those at Pedro Miguel by 2. How many gates at Gatun? At all three places?

17. Contracts for construction of the gates of the Panama Canal were awarded to the lowest bidder, whose bid was $728,000 under his nearest competitor. If the sum of both bids was $11,477,000, what was the contract price?

18. A rectangle has the same area as another one 10 feet longer and 6 feet narrower. It also has the same area as a rectangle 4 feet longer and 3 feet narrower. Find the dimensions of the rectangle.

19. If the length and width of a rectangle be increased each by one foot, the area will be increased by 18 square feet. But if the length alone be decreased one foot, the area will be decreased by 2 square feet. What are its dimensions?

20. A rifle ball was fired at a target 3300 feet away, and 5 seconds later the man who fired the shot heard it strike the target. If the speed of the bullet was 1500 feet per second, what was the rate at which the sound of the bullet striking the target traveled back to the marksman?

21. A traveller finds that he has 6 hours to wait at a station and decides to use it by taking a trip into the country. He walks out at the rate of three miles an hour and rides back in an automobile at the rate of 15 miles an hour. How far may he go?

22. On December 21st, the night of Hammerfest, Norway, is 13 hours and 30 minutes longer than the day. At what hour does the sun rise there?

23. In physics t designates the number of seconds required by a pendulum to swing from one position of rest to another, through a small angle, and it is found from the relation, $t = \pi\sqrt{l / g}$. If $g = 32.15$, $\pi = 3.1416$, and $l = 3$ feet, find the value of t. If $t = 1$ second, g and π having the same values as before, find l, that is, the length of a pendulum that beats seconds.

24. On a Centigrade thermometer (see Fig. 22) the freezing and boiling points are marked 0° and 100° respectively; and on a Fahrenheit thermometer they are marked 32° and 212° respectively. Between these two fixed points there are 100 degrees on the Centigrade thermometer and 180 on the Fahrenheit. The reading of one thermometer being known that of the other may be found from the relation, $F = 32° + \frac{9C}{5}$. (1). From (1) show that $C = \frac{5}{9}(F - 32)$. (2).

25. Using Equations (1) and (2) of the preceding example transform the following Fahrenheit readings into Centigrade: 41°; 75°; − 30°; 180°; 270°; − 36°; − 100°.

26. Using the same equations as in Example 25, transform the following Centigrade readings into Fahrenheit: 18°; 45°; 180°; − 16°; − 80°.

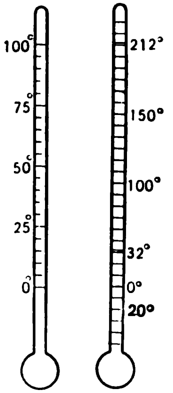

FIG. 22

27. As seen from the earth the sun and moon, moving in the same direction, complete their respective circuits of the heavens in approximately $365\frac{1}{4}$ days and 28 days. Supposing their motion to be uniform, how long is it from one phase of the moon to another exactly similar phase; say from one full moon to the next?

(Suggestion: Let x = the number of days from one phase of the moon to another similar phase. Then it is shown in astronomy that

$$\frac{1}{28}-\frac{1}{365}=\frac{1}{x}.\Big)$$

28. In the center of a pond 10 feet square, a reed projected 1 foot above the surface; when the top of this reed was pulled over to the bank it just reached the edge of the surface. How deep was the water? (From a Chinese arithmetic of 2000 B.C.)[*]

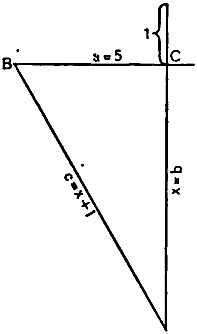

FIG. 23

[†] (Suggestion. Let x = the depth of the water in feet. Then in Fig. 23, $x = AC$, $x + 1 = AB$, $5 = BC$; and $(x+1)^2 = x^2 + 5^2$.)

29. In a pond the top of a lotus bud reached 6 inches above the surface; but blown by the wind it just reached the surface at a point 2 feet from its upright position. How deep was the water? (From the works of a Hindu writer, 1150 A.D.)[*]

30. Find the number, $\frac{1}{3}$ of which plus 1. multiplied by $\frac{1}{4}$ of which plus 2, equals the number plus 13, (From Mohammed ben Musa Alhowarazmi, a Persian mathematician of 800 A.D., from the title of whose book the word Algebra is derived.)[*]

[*] From Klein's Famous Problems in Geometry, translated by Beman and Smith.

[†] The solution of this and the following example is based upon the following relation between the sides of a right-angled triangle: In any right triangle the square on the side opposite the right angle equals the sum of the squares on the other two sides. Thus, in Fig. 23 above $c^2 = a^2 + b^2$. In geometry this is known as the Pythagorean theorem, from the name of Pythagoras, the Grecian mathematician who first proved it.

31. A horse and a donkey laden with corn, were walking together. Said the horse to the donkey: "If you gave me one measure of corn, I should carry twice as much as you; but if I gave you one we should carry equal burdens." Tell me their burdens, O most learned master of geometry. (Attributed to Euclid, about 300 B.C.)*

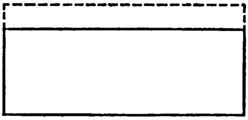

FIG. 24

32. The length of a football field is 140 feet greater than its width. If a strip 20 feet wide be added to each end and each side the area will be increased by 18,400 square feet. Find the dimensions of the field.

33. A rectangular piece of ground is 13 feet longer than it is wide. If 2 feet were added to its width while its length remains unchanged its area would be increased by 40 square feet. Find the length of the two sides. (Fig. 24.)

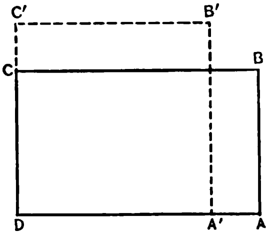

FIG. 25

34. A rectangle is twice as long as it is wide. If 5 feet were taken from its length and added to its width the rectangle would be square. What are its dimensions? (Fig. 25.)

35. The width of a room is $\frac{3}{5}$ of its length. If 3 feet were taken from the length and the same amount added to its width the room would be square. What are its dimensions? (Fig. 26.)

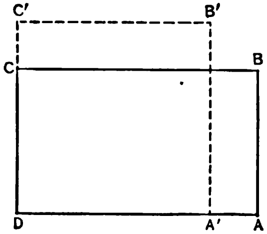

FIG. 26

36. At a school entertainment the prices of admission were fixed at 50 cents for adults, and 35 cents for children. The number of paid admissions was 100, and the receipts were $39.50. How many children were included in the 100 who paid admission?

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* From Klein's Famous Problems in Geometry, translated by Beman and Smith.

37. A rectangular field has the same area as each of two other fields, one of which is 6 rods longer and 2 rods narrower than the given field, while the other is 3 rods shorter and 2 rods wider than the given field. Find the dimensions of the given field.

38. Two trains, one twice the length of the other, are running on parallel tracks. The rate of the longer train is 44 feet per second, while that of the shorter is 66 feet per second. When going in opposite directions it takes them 15 seconds to pass. Find the length of each train.

39. Two trains run on parallel tracks. One of the trains is 350 feet longer than the other and travels at a rate of 30 miles per hour, while the shorter train makes 45 miles per hour. When going in the same direction it takes $1\frac{1}{4}$ minutes for the two trains to pass each other. What is the length of each train?

40. A dairyman has milk cans of three different sizes. He finds that the quantity of milk he has will fill all the cans once; the smallest can four times; the second largest can three times with 8 gallons to spare; and the largest can twice with 4 gallons to spare. Find the capacity of each can.

41. The California Limited, when four hours out of Chicago, met with an accident and was delayed one hour. Resuming its westward journey at $\frac{3}{4}$ its former rate, it arrived in Kansas City 3 hours late. If the accident had occurred 140 miles farther west the train would have reached Kansas City only 2 hours late. Find the usual rate of this train between Chicago and Kansas City and the distance between these two cities.

42. It takes A three hours longer to walk from P to Q than it does B, the distance being 15 miles. But, if A should double his rate of walking and B continue at the same rate as before, A would make the trip in one hour less than B. What are their rates of walking?

43. If the sides of a rectangular field were increased each by 5 rods, the area would be 2 acres. If the length were increased by 5 rods and width decreased by 7 rods, the area would be one acre. Find the dimensions of the field.

44. A farm consisting of 200 acres is 40 rods longer than it is wide. The distance around the farm is $2\frac{1}{4}$ miles. What are its dimensions?

45. In triangle ABC (Fig. 27), $a = c$ and b is 10 less than a. If the sum of a, b, and c is 80, find each side.

46. In the same triangle ABC (Fig. 27), $a + c = 60$; $a + b = 50$; and $b + c = 50$. Find a, b, and c from these equations.

47. The sum of the areas of two rectangles is 65 and the difference is 35. If the width of one is 5 and of the other 3, what are their lengths?

48. The sum of the three sides a, b, and c of a certain triangle is 60. If a is 4 more than twice c, and b and c together are 4 more than a, how long are the sides a, b, and c?

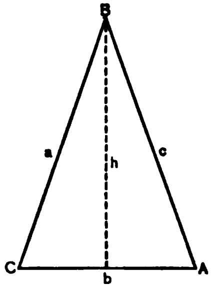

FIG. 27

49.* The average age of three persons is 27 years. The average age of the first and second is 38 years and of the second and third $16\frac{1}{2}$ years. What are their ages?

50. Find each of the angles of a triangle, if the sum of the first and second is twice the third, and the sum of the first and third equals the second.

79. Dependent Linear Equations.

Example. Given: $x - 2y = 4$

$$3x - 6y = 12.$$

Solve for x and y.

In attempting to construct the graphs for these equations it is found that both represent the same line, MN.

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* Problems 49 and 50 involve three unknowns. For the present we leave it to the originality of the pupil to devise a method of solving a system of three equations.

(Fig. 28.) From this fact one would infer that, instead of having two independent equations in the foregoing example, there is really but one equation. Referring to the given equations it will be seen that the second equation may be derived from the first by multiplying both sides of the first by 3; also that the first may be derived from the second by dividing both sides of the second by 3.

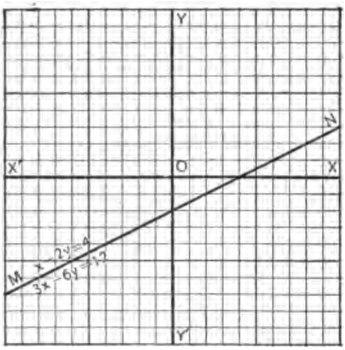

FIG. 28

Equations that may be derived thus from other equations are said to be *dependent equations.*

Throughout the solution of systems of simultaneous linear equations by elimination as in § 77, new equations are continually being derived from others either by addition, subtraction, multiplication, or division. All such equations thus derived are dependent equations.

80. Equivalent Equations. In the solution of a linear equation it has been observed that:

(1) The same number may be added to both sides.
(2) The same number may be subtracted from both sides.
(3) Both sides may be multiplied by the same number.
(4) Both sides may be divided by the same number.

An equation derived from a given equation by performing upon the latter any one or all of the four processes named above is equivalent to the given equation, provided the given equation is not multiplied or divided either by zero or by an expression containing the unknown number.

Thus (1) $\frac{2}{3}x + 1 = 6$,
and (2) $2x + 3 = 18$ are equivalent equations, since (2) is obtained from (1) by multiplying (1) by 3.

Solving (1) for x, gives $x = 7\frac{1}{2}$.

And solving (2) for x, gives $x = 7\frac{1}{2}$ also.

Furthermore, there is no other solution for either.

Hence, *two equations are said to be equivalent when every solution of one is also a solution of the other.*

On the other hand (1) $\frac{2}{3}x + 1 = 6$,
and (2) $\frac{2}{3}x^2 + x = 6x$, are not equivalent, as will be shown later, since (2) is derived from (1) by multiplying (1) by x, the unknown number.

81. Inconsistent Equations.

Example. Let $x + y = 5$

$2x + 2y = 20$, be the two linear equations whose solution is desired.

Constructing the graphs for these two equations it will be found that they represent a pair of parallel lines. Hence there is no point of intersection and no set of values for x and y that satisfies both equations. (Fig. 29.)

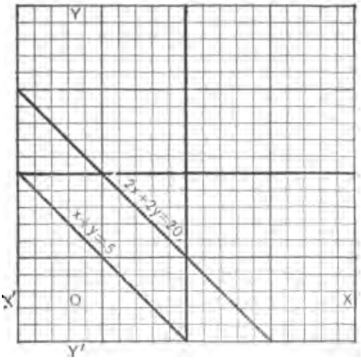

FIG. 29

Referring to the given equations, and dividing both sides of the second equation by 2, we have $x + y = 5$ and $x + y = 10$ as the two equations. Clearly the sum of x and y could not be 5 and 10 at the same time if x and y have the same values in both equations. Such equations as those given in the above example are called *inconsistent or incompatible equations.*

EXERCISE

82. Determine the nature of the following systems of linear equations by constructing their graphs:

1. $\begin{cases} x + y = 4, \\ x + y = 10. \end{cases}$

2. $\begin{cases} 2x + y = 5, \\ 4x + 2y = 12. \end{cases}$

3. $\begin{cases} x - 2y = 1, \\ 2x - 4y = 4. \end{cases}$

4. $\begin{cases} 4x + 8y = 20, \\ 2x + 4y = 10. \end{cases}$

5. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = 2, \\ 2x - y = 2. \end{cases}$

6. $\begin{cases} x - y = 3, \\ 3x - 9y = 9. \end{cases}$

7. $\begin{cases} 2x + 3y = 12, \\ 4x + 6y = 16. \end{cases}$

8. $\begin{cases} x - y = 1, \\ 5x - 5y = 10. \end{cases}$

9. $\begin{cases} x - \frac{1}{2}y = 0, \\ 2x - y = 1. \end{cases}$

10. $\begin{cases} x + y = -2, \\ 2x + 2y = -4. \end{cases}$

MISCELLANEOUS EXERCISES FOR REVIEW

83. Give results to the following at sight:

1. $3(-8)$.

2. $-5a(10ac^3)$.

3. $(-5a^3)^2$.

4. $(-2a)(-3ab)^2$.

5. $-63 \div -7$.

6. $-12xy - (-4xy)$.

7. $-18abc + 6abc - 8abc - 5(-2abc)$.

8. $(7y)(5y - 4y)$.

9. $(-25xy^2z + 15x^2y^2z) \div (-5xy)$.

10. $8 \cdot 0 - 5 + \frac{9}{4}$.

11. Take $2a - 5c$ from 0.

12. Add $-(a-b)$, $2(b-a)$, and $-3(a-b)$.

13. $-bc \cdot b^2 \cdot c^2 \cdot d^2$.

14. $(-4d)(5cd)(-3ad)$.

15. Multiply 49 by 29.

Suggestion. $49 = (50 - 1)$

$$29(50-1) = 29 \times 50 - 29 \times 1 = 1450 - 29 = 1421.$$

16. In like manner multiply 99 by 39; 51 by 29; 48 by 33; 52 by 44; 61 by 24.

Perform operations indicated:

17. $\left(\frac{8a^2}{9} - \frac{a}{3} - \frac{3}{4}\right) \div \left(\frac{2a}{3} + \frac{1}{2}\right)$.

18. $\left(\frac{8x^3}{27} + \frac{27}{64}\right) \div \left(\frac{2x}{3} + \frac{3}{4}\right)$.

19. $(a^{2x} + b^{2n} + 2b^nc^x - c^{2x}) \div (a^x + b^n + c^x)$. Check for $a = b = c = n = x = 1$.

20. $\left(\frac{2x}{3} - ax + \frac{2a}{3}\right)\left(\frac{3x}{4} + \frac{ax}{4} + \frac{a}{3}\right)$.

21. $(2.4x^3 - .12x^2y + 4.32y^3) \div (1.5x + 1.8y)$.

Simplify the following:

22. $(x+y)(y+z) - (z+u)(u+x) - (x+z)(y-u)$.

23. $(y+a)(y+b) - (a+b+c)(y+a) + a^2 + ab + b^2 + 3ay$.

24. $(a+b)^2 - (a+b)(a-b) - a(2b-2) - (b^2 - 2a)$.

25. $a(b-c) - b(a-c) + c(a-b) - (a-b)(a-c)(b-c)$. Check for $a = b = c = 1$.

Evaluate the following for $y = 2$, $a = 3$, $b = 1$:

26. $y^{a+b} \cdot y^{a-b}$.

27. $(y^a)^2 \div (y^{a-b})$.

28. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b - c$. Check for $a = b = c = 1$.

29. Solve for y: $.3y - 4 = .2y + .5$.

30. Multiply $3.2x^2 - 4.5xy + 1.8y^2$ by $1.5x + 3.5y$. Check for $y = 1 = x$.

CHAPTER VI

FACTORING—QUADRATIC EQUATIONS

(a) Factoring

84. The Product of the Sum and the Difference of Two Numbers. From the multiplication of $(a + b)$ by $(a - b)$, as given below:

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

we have the formula,

VIII $$(\mathbf{a} + \mathbf{b})(\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2.$$

Check. $(a + b)(a - b) = (2 + 1)(2 - 1),$
if $a = 2$ and $b = 1$. $= 3 \cdot 1$
$= 3.$

But $a^2 - b^2 = 4 - 1 = 3.$

Therefore, since the product of the numerical values equals the numerical value of the product, we conclude that the result is correct.

In words, Formula VIII means that *the product of the sum and the difference of the same two numbers equals the difference of their squares.*

ORAL EXERCISE

Perform the multiplications indicated below by inspection and check results.

1. $(x + y)(x - y)$.
2. $(a - 5)(a + 5)$.
3. $(3c - 9)(3c + 9)$.
4. $(4m - 3np)(4m + 3np)$.
5. $(a^2c - 9m^2)(a^2c + 9m^2)$.
6. $(a^x + b^y)(a^x - b^y)$.
7. $(3x^c - 5y^n)(3x^c + 5y^n)$.
8. $(a - b + c)(a + b - c)$.
9. $[2(x + y) - 3p][2(x + y) + 3p]$.
10. (*a*) $(47)(33)$.

    $47 = 40 + 7$ and $33 = 40 - 7$.

    (*b*) $(84)(76)$.

85. Factors of the Difference of the Squares of Two Numbers. We have already learned that the square of any number is obtained by multiplying it by itself or using it twice as a factor. Consequently, the square root of any number is one of its two equal factors. The radical sign, $\sqrt{\ }$, when placed over a number, indicates that its square root is to be taken.

Thus, since $2 \times 2 = 4$,[*] $\sqrt{4} = 2$.

And since $3a \times 3a = 9a^2$, $\sqrt{9a^2} = 3a$.

From Formula VIII we have

$$\text{VIII}' \qquad \mathbf{a}^2 - \mathbf{b}^2 = (\mathbf{a} + \mathbf{b})(\mathbf{a} - \mathbf{b}).$$

Hence, $4a^2 - 9b^2 = (2a + 3b)(2a - 3b)$. (Check as in § 66).

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[*] It is also true that $-2 \times -2 = 4$ and hence that the $\sqrt{4} = -2$. But for the present purposes we shall consider only the positive square roots.

This means that *the factors of the difference of the squares of two numbers consist of two binomials, one being the sum and the other the difference of the numbers.*

EXERCISE

Factor the following:

1. $100\,t^2 - 144\,u^2$. **2.** $(a - b)^2 - 9$.

3. $(n + v)^2 - (m - n)^2$. **4.** $9(x^2y^2 - m^4n^4)$.

5. $(a^2 + b^2)^2 - 81$. **6.** $4(x + y)^2 - 9(a - b)^2$.

7. $9\,a^8 - \frac{1}{4}\,a^2$. **8.** $(25\,b^2 - 36\,c^2)^2 - 81\,a^4$.

9. $x^8 - y^8$. (Four factors.) **10.** $a^{4m} - b^{6n}$.

11. 99. $(99 = 100 - 1.)$ **12.** $16\,x^{2m} + {}^{2} - 64\,y^{4n}$.

13. 621. $(621 = 625 - 4.)$ **14.** $4\,b^{4m} - 9\,c^{2m^2}$

15. $4(a + b)^2 - 1$. **16.** $a^{16} - 4\,b^{16}$. (Five factors.)

17. $1 - (2\,x + 3\,y)^2$. **18.** $(a^2 - b^2)^2 - c^4$.

Solution: $(a^2 - b^2)^2 - c^4 =$
$(a^2 - b^2 - c^2)(a^2 - b^2 + c^2)$.

19. $81 - (a - b)^4$. **20.** $256 - x^4y^8$.

86. Product of Binomials Having a Common Term.

Example. Multiply $(x + 3)$ by $(x + 5)$.

$$\begin{array}{l} x + 3 \\ x + 5 \\ \hline x^2 + (3 + 5)x + 15 = x^2 + 8\,x + 15. \end{array}$$

Check. Let $x = 1$. $(x + 3)(x + 5) = (1 + 3)(1 + 5) = 4 \cdot 6 = 24$.
And $x^2 + 8\,x + 15 = 1 + 8 + 15 = 24$.

Observe that (1) x^2 is the square of the common term x:
(2) $8\,x$ is the product of x and the sum of 3 and 5.
(3) 15 is the product of 3 and 5.

In general,

IX $$(\mathbf{x} + \mathbf{a})(\mathbf{x} + \mathbf{b}) = \mathbf{x}^2 + (\mathbf{a} + \mathbf{b})\,\mathbf{x} + \mathbf{ab}.$$

This means that *the product of two binomials having a common term equals the square of the common term plus the product of the common term and the algebraic sum of the other two terms, plus also the product of the other two terms.*

EXERCISE

In accordance with formula IX give the products in the following by inspection. Check results.

1. $(x + 5)(x + 6)$.
2. $(x - 3)(x + 7)$.
3. $(-x - 4)(-x - 8)$.
4. $(2x - 3)(2x - 5)$.
5. $(y + a)(y - 3a)$.
6. $(m - 2a)(m + a)$.
7. $(a^2 - 7)(a^2 - 1)$.
8. $(3b - c)(3b + 2c)$.
9. $(4m - a)(4m - 8a)$.
10. $(2c^2 - 3d)(2c^2 + 7d)$.
11. $(x - m)(x - n)$
12. $(y - 2p)(3y - 2p)$.
13. $(x^2 - bc)(x^2 + 3bc)$.
14. $(2a - b)(2a + c)$.
15. $(36)(39)$.

Suggestion: $36 = (30 + 6)$, $39 = (30 + 9)$.
Hence, $36 \times 39 = (30 + 6)(30 + 9)$.

16. $(107)(105)$.
17. $(54)(57)$.
18. $(x^{2p} - 3y)(x^{2p} + 6y)$.
19. $(99)(103)$.
20. $(x^a - 2y)(x^a + 4y)$.

87. Factors of the Quadratic Trinomial, $\mathbf{x}^2 + \mathbf{bx} + \mathbf{c}$.

From Formula IX we have

IX′ $$\mathbf{x}^2 + (\mathbf{a} + \mathbf{b})\mathbf{x} + \mathbf{ab} = (\mathbf{x} + \mathbf{a})(\mathbf{x} + \mathbf{b}).$$

Example 1. $x^2 + 5x + 4 = (x + 4)(x + 1)$.

For, 4 is the product of two numbers, $+4$ and $+1$, whose sum is $+5$. It should be observed also that 2 and 2, as well as 4 and 1, are factors of 4; but 2 and 2 added together do not make 5, the coefficient of the middle term.

Hence, $(x + 4)$ and $(x + 1)$ are the factors of $x^2 + 5x + 4$.

Example 2. $x^2 - 5x + 4 = (x - 4)(x - 1)$.

For the two factors of $+4$ whose sum is -5 are -4 and -1. -2 and -2, while factors of $+4$, will not do, since the sum of -2 and -2 is not -5.

Hence, $(x - 4)$ and $(x - 1)$ are the factors of $x^2 - 5x + 4$.

Example 3. $x^2 + 5x - 24 = (x + 8)(x - 3)$.

For, of all the factors of -24, the two whose sum is $+5$ are $+8$ and -3.

Hence, $(x + 8)$ and $(x - 3)$ are the factors of $x^2 + 5x - 24$.

Example 4. $x^2 - 5x - 24 = (x - 8)(x + 3)$.

For, of all the factors of -24, the two whose sum is -5 are -8 and $+3$.

Hence, $(x - 8)$ and $(x + 3)$ are the factors of $x^2 - 5x - 24$.

From the foregoing examples it is evident that *the factors of the quadratic trinomial,** $x^2 + bx + c$, *consist of two binomials having a common term* x *and whose other two terms are those factors of* c *whose algebraic sum is* b.

EXERCISE

Factor and check the following:

1. $a^2 - 7a + 12$.

2. $a^2 - 7a - 18$.

3. $a^2 + 9a - 10$.

4. $x^2 - 12ax - 13a^2$.

5. $R^2 - 7pR + 10p^2$.

6. $m^2 - 32m + 175$.

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* Unless two numbers exist whose product is c and whose algebraic sum is b this trinomial cannot be factored.

7. $m^2 - 17\,mn + 30\,n^2$. **8.** $c^2 + 28\,bc + 192\,b^2$.
9. $a^2b^2 + 2\,abc - 3\,c^2$. **10.** $a^4 - 17\,a^2b^2 + 60\,b^4$.
11. $1 - 5\,x - 6\,x^2$. **12.** $s^2 + 30\,st - 175\,t^2$.
13. $a^{2m} - 10\,a^mb^2 - 39\,b^4$. **14.** $r^{4x} + 12\,r^{2x}v^n + 27\,v^{2n}$.
15. $(x - y)^2 - 11\,(x - y) - 60$.
16. $m^4 - 11\,m^2n^2 + 24\,n^4$.
17. $p^2q^2r^2 - 15\,pqrx^2 - 54\,x^4$. **18.** $a^2 - ab - 2\,b^2$.
19. $a^2 + 3\,ab + 2\,b^2$. **20.** $a^2 + ab - 2\,b^2$.

88. Product of Two Binomials Having Corresponding Terms Similar.

Example. $(3\,x + 5)(x - 4) = 3\,x^2 - 7\,x - 20$.

For, $3\,x^2$ and -20 are products obtained by multiplying together the similar terms $3\,x$ and x, 5 and -4; $-7\,x$ is obtained by adding the products resulting from the multiplication of the dissimilar terms, $3\,x$ and -4, x and 5.

Usual Method of Multiplication

$$\begin{array}{l}
3\,x + 5 \\
x - 4 \\
\hline
3\,x^2 + 5\,x \\
 - 12\,x - 20 \\
\hline
3\,x^2 - 7\,x - 20
\end{array}$$

Cross Product Method of Multiplication

$$\begin{array}{l}
3\,x + 5 \\
x - 4 \\
\hline
3\,x^2 + (-12 + 5)\,x - 20 = \\
3\,x^2 - 7\,x - 20.
\end{array}$$

Check.
$$\begin{aligned}
(3\,x + 5)(x - 4) &= (9 + 5)(3 - 4), \text{ if } x = 3, \\
&= (14)\,(-1) = -14. \\
3\,x^2 - 7\,x - 20 &= 27 - 21 - 20 = -14. \\
& -14 = -14.
\end{aligned}$$

Therefore, since the product of the numerical values is the same as the numerical value of the product, we conclude that the result is correct.

From the foregoing it will be seen that *the product of two binomials which have corresponding terms similar equals the sum of the products obtained first, by multiplying the similar terms together; second, by multiplying the dissimilar terms together and adding these results.*

EXERCISE

Perform the operations indicated and check results.

1. $(3a - 7)(a + 9)$.
2. $(x^2 - 5)(2x^2 + 3)$.
3. $(4m^2 - 7)(-2m^2 - 8)$.
4. $(-a^2b + 10)(3a^2b - 6)$.
5. $(7pq - 3xy)(3pq - 4xy)$.
6. $(-ab^2 + 3cn)(7ab^2 + 2cn)$.
7. $(10x^2 - 4y^2z)(x^2 + 12y^2z)$.
8. $(-a^2 - b^2)(-2a^2 + 2b^2)$.
9. $(3a^x - 2b)(4a^x + 3b)$.
10. $(-2a^p - 3c)(-2a^p + 5c)$.

89. Factors of Quadratic Trinomial, $ax^2 + bx + c$. It will be seen from § 88 that the quadratic trinomial, $ax^2 + bx + c$, may be factored into two binomials when a and c can be resolved into two sets of factors such that the sum of their cross products will be equal to b.

We shall consider four cases based upon the signs of the coefficients, a, b, and c.

CASE I. ALL SIGNS POSITIVE.

Example 1. Factor $3a^2 + 13a + 4$.

Two sets of factors of $3a^2$ and 4 are to be found such that the sum of their cross-products as indicated below shall be $+ 13a$.

$$\begin{array}{r} 3a + \\ \underline{a + 4} \end{array}$$

Here, $3a \times a = 3a^2$, and $4 \times 1 = 4$.

Cross-products, $3a \times 4 + a \times 1 = 12a + a = 13a$.

Hence, $3a^2 + 13a + 4 = (3a + 1)(a + 4)$.

Several trials will often be necessary before the factors are found. For instance, if we had taken as trial factors, $(3a + 4)$ and $(a + 1)$,

$$\begin{array}{c} 3a + 4 \\ \times \\ \underline{a + 1} \end{array}$$

As shown above, these would have given the first and last terms of $3a^2 + 13a + 4$, but the middle term would have been the sum of $3a \cdot 1$ and $a \cdot 4$ or $3a + 4a$, which equals $7a$.

Hence, $(3a + 4)$ and $(a + 1)$ cannot be factors of $3a^2 + 13a + 4$.

CASE II. MIDDLE TERM NEGATIVE, OTHER TERMS POSITIVE.

Example 2. Factor $3a^2 - 10a + 3$.

Factors of $3a^2$ and 3, the sum of whose cross-products is $-10a$ are found to be $3a - 1$ and $a - 3$.

$$\begin{array}{r} 3a - 1 \\ \underline{a - 3} \\ -9a \\ \underline{-a} \\ -10a \end{array}$$

Hence, $3a^2 - 10a + 3 = (3a - 1)(a - 1)$.

CASE III. FIRST TERM POSITIVE, OTHER TERMS NEGATIVE.

Example 3. Factor $4a^2 - 4ac - 3c^2$.

Factors of $4a^2$ and $-3c^2$, the sum of whose cross-products is $-4ac$ are $2a$, $3c$, $2a$ and c.

$$\begin{array}{r} 2a - 3c \\ \underline{2a + c} \\ +2ac \\ \underline{-6ac} \\ -4ac \end{array}$$

Hence, $4a^2 - 4ac - 3c^2 = (2a - 3c)(2a + c)$.

CASE IV. FIRST AND SECOND TERMS POSITIVE, LAST TERM NEGATIVE.

Example 4. Factor $4a^2 + 7a - 2$.

Factors of $4a^2$ and -2, the sum of whose cross-products is $+7a$ are $4a - 1$ and $a + 2$.

$$\begin{array}{r} 4a - 1 \\ \underline{a + 2} \\ +8a \\ \underline{-\ \ a} \\ +7a \end{array}$$

Hence, $4a^2 + 7a - 2 = (4a - 1)(a + 2)$.

Based upon the foregoing examples certain observations will now be stated for the purpose of simplifying the work of factoring the quadratic trinomial, $ax^2 + bx + c$.

(1) *If all signs are positive, only positive factors need be tried.*

(2) *If the first and last terms are positive and the middle term negative only negative factors of the last term should be tried.*

(3) *If the last term is negative, one of its factors must be plus and the other minus.*

EXERCISE

Factor and check:

1. $2a^2 + 6a + 4$.

2. $4x^2 + 7x - 15$.

3. $5x^2 - 2x - 7$.

4. $9a^2 - 6a - 120$.

5. $15m^2 - 77m + 10$.

6. $4a^2 + 10ab + 4b^2$.

7. $12c^2 - 23c + 10$.

8. $2m^2 - 3mn - 2n^2$.

9. $4y^2 - y - 14$.

10. $24a^2 + 42ab - 45b^2$.

11. $25a^2 - 101ab + 4b^2$.

12. $60x^2y^2 - 59xyz - 20z^2$.

13. $30a^2 + 95a - 35$.

14. $80r^2 - 240r + 180$.

15. $44x^2 + 25x - 91$. **16.** $36a^{2x} + 45a^x - 91$.
17. $56x^{2a} - 113x^a + 56$. **18.** $9(a+b)^2 - 6(a+b) - 120$.
19. $5(x-y)^4 - 2(x-y)^2 - 7$. **20.** $16a^{2x+2} - 44a^{x+1} + 30$.

90. Factors Determined by Division.

Example 1. Since $(a^3 + b^3) \div (a+b) = a^2 - ab + b^2$, it follows that

X $$\mathbf{a^3 + b^3 = (a+b)(a^2 - ab + b^2)},$$

and hence that $(a+b)$ and $(a^2 - ab + b^2)$ are the factors of $(a^3 + b^3)$.

Example 2. Since $(a^3 - b^3) \div (a-b) = a^2 + ab + b^2$, it follows that

XI $$\mathbf{a^3 - b^3 = (a-b)(a^2 + ab + b^2)},$$

and hence that $(a-b)$ and $(a^2 + ab + b^2)$ are the factors of $(a^3 - b^3)$.

Example 3. Likewise $(a^5 \pm b^5)$ [*] $\div (a \pm b) = a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4$.

Therefore, $a^5 \pm b^5 = (a \pm b)(a^4 \mp a^3b + a^2b^2 \pm ab^3 + b^4)$, and $(a \pm b)$and $(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4)$ are the factors of $(a^5 \pm b^5)$.

Example 4. $(a^9 \pm b^9) \div (a^3 \pm b^3) = a^6 \mp a^3b^3 + b^6$.

Therefore, $a^9 \pm b^9 = (a^3 \pm b^3)(a^6 \mp a^3b^3 + b^6)$ and $(a^3 \pm b^3)$ [†] and $a^6 \mp a^3b^3 + b^6$ are factors of $(a^9 \pm b^9)$.

Observations as to the foregoing results:

1. In each of these examples, *the number of terms* in the quotient is determined by the *number of times* the exponent

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[*] Read a^5 plus or minus b^5. Using the double sign as here makes it possible to state two problems together. Read upper signs all through for $(a^5 + b^5)$ and lower signs for $(a^5 - b^5)$.

[†] $(a^3 \mp b^3)$ may be further factored as in Examples 1 and 2.

of the first term of the dividend contains the exponent of the first term of the divisor. Thus, in Example 3, exponent 5 contains exponent 1 five times; therefore, five terms will be found in the quotient. In example 4 the exponent 9 contains the exponent 3 three times ; hence, there are three terms in the quotient.

2. In each of the above cases the *exponent* of the first term in the quotient is the *difference* between the exponent of the first term of the dividend and the exponent of the first term of the divisor. In each of the succeeding terms the *exponent* of the first number is obtained by *subtracting* from the exponent of that number in the preceding term the exponent of the first term in the divisor.

Thus, in Example 3, the exponent 3, of a in the term a^3b, is the difference between 4, the exponent of a in the preceding term and 1, the exponent of a in the divisor. The exponent 2 of a in the term a^2b^2 is found in like manner by subtracting 1 from 3. In each of the other terms that follow the exponent of a is 1 less than in the preceding term. In like manner in Example 4, the exponent 6 of the first term of the quotient is obtained by subtracting 3, the exponent of a, in the first term of the divisor, from 9, the exponent of a, the first term of the dividend. In each of the succeeding terms the exponent of a is 3 less than in the term next preceding.

3. The second term of the divisor appears in the quotient for the first time in the second term of the quotient and there has the *same exponent* that it had in the divisor. In each of the succeeding terms its exponent is found by *adding* its exponent in the divisor to its exponent in the next preceding term of the quotient.

4. In dividing the sum of odd powers by the sum of the numbers, *signs of the quotient alternate* from + to − and from − to +. In dividing the difference of odd powers

by the difference between the numbers, *the signs of all the terms in the quotient are* $+$.

From the foregoing we have the general formula:

XII $$\mathbf{a^n \pm b^n = (a \pm b)(a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp \ldots + b^{n-1}),}$$ **n being odd.**

In words this means that:

(1) *The sum or the difference of the same odd powers of two numbers may be divided by the sum or the difference of the numbers.*

(2) *The quotient thus obtained and the divisor constitute the factors of the given binomial.*

EXERCISE

Write out the quotients in the following by inspection and in accordance with the foregoing observations. Check.

1. $(x^3 + y^3) \div (x + y)$.

2. $(m^3 - n^3) \div (m - n)$.

3. $(m - n)^5 \div (m - n)$.

4. $(a^7 + b^7) \div (a + b)$.

5. $(x^9 - y^9) \div (x - y)$.

6. $\left(\frac{m^3}{8} + \frac{n^3}{27}\right) \div \left(\frac{m}{2} + \frac{n}{3}\right)$.

7. $(1 - x^3) \div (1 - x)$.

8. $(a^5 + 32) \div (a + 2)$.

9. $(243 - c^5) \div (3 - c)$.

10. $(m^3 - 27) \div (m - 3)$.

11. $(y^6 + m^6) \div (y^2 + m^2)$.

12. $(32 + m^5) \div (2 + m)$.

Suggestion: $y^6 = (y^2)^3$. Hence, $y^6 + m^6 = (y^2)^3 + (m^2)^3$.

Hence $y^6 + m^6$ may be considered as the sum of odd powers.

What binomial divisor or factor has each of the following, and how many terms in each quotient?

13. $x^3 - 8$.

14. $a^5 - b^5$.

15. $27\,a^3 - 216\,c^3$.

16. $x^3y^6 - z^3$.

17. $m^{10} + n^{10}$.

18. $a^7 + b^7$.

19. $64\,a^6 + 125\,b^6$. 20. $\frac{m^3}{8} + \frac{n^3}{64}$.

21. $243\,c^5 - 32\,y^{10}$. 22. $1 - x^3$.

23. $x^5 - 1$. 24. $8\,a^3 - \frac{1}{27}$.

25. $x^6 - 125$. 26. $\frac{x^3}{27} - \frac{y^6}{64}$.

27. $a^3b^3 + 1$. 28. $512\,c^6 + 343\,d^3$.

29. $8\,a^{15} + 27\,b^{15}$. 30. $729\,m^{12} + 8\,n^{3n}$.

31. $a^{3m} - 8\,c^{3n}$. 32. $32\,x^{15} + 243\,y^{15}$.

33. $a^{5c} + b^{10c}$. 34. $a^5 - 32\,t^5$.

35. $R^7 - 1$. 36. $8\,a^3 - (m + n)^3$.

37. $(a - b)^3 - 27$. 38. $a^{6c} - b^{9d}$.

39. $a^6 - x^6$. Factor in two ways. 40. $a^{3x} + b^{6y}$.

91. The Sum of the Same Even Powers of two numbers is divisible by the *sum* of the numbers *only* in the case where such even powers may be considered as *odd* powers of smaller even powers. Thus, a^6 may be written as $(a^2)^3$ and a^{10} as $(a^2)^5$; but a^4 may not be so written.

Hence, $(a^6 + b^6) \div (a^2 + b^2) = a^4 - a^2b^2 + b^4$,
and

$$(a^{10} + b^{10}) \div (a^2 + b^2) = a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8.$$

But, $(a^4 + b^4)$ may not be divided exactly by any binomial.

92. Involution is the process of raising a number to a required power. It means that a number or expression is to be multiplied by itself or used as a factor a certain number of times as indicated by the exponent.

Thus, 4^3 means $4 \times 4 \times 4$, as already noticed in paragraph 14. In like manner $(2\,a^2b)^3$ means $2\,a^2b \times 2\,a^2b \times 2\,a^2b$, which equals $8\,a^6b^3$. (By § 70.)

Check. Let $a = b = 1$. Then $2\,a^2b = 2 \times 1 \times 1 = 2$ and $(2\,a^2b)^3 = 2^3 = 8$.

But $8a^6b^3 = 8 \times 1 \times 1 = 8$.

Therefore, since the cube of the numerical value equals the numerical value of the cube, the result is probably correct.

Hence, a monomial may be raised to any desired power by the ordinary rules for multiplication of monomials.

Referring again to the expression $(2a^2b)^3 = 8a^6b^3$, it should be noticed that

(1) 8, the coefficient of the power, is the *cube* of 2, the coefficient of the given term.

(2) 6, the exponent of a in the power, is *three* times 2, the exponent of a in the term.

(3) 3, the exponent of b in the power, is *three* times 1, the exponent of b in the term.

Again, since an odd number of negative factors produces a negative product, and an even number of negative factors produces a positive product, we have the following rule for the involution of a monomial:

(1) *Raise the coefficient of the given term to the power indicated.*

(2) *Multiply the exponent of each letter by the number indicating the power.*

(3) *Prefix a minus sign only in case the term itself is minus and the power odd. In all other cases the sign will be plus.*

EXERCISE

Perform the operations indicated:

1. $(3a^3b^2c^4)^3$.

2. $\left(\frac{-5mnp}{2}\right)^4$.

3. $(-2x^2y^3)^4$.

4. $-(9a^4b)^2$.

5. $(-5m^2n)^3$.

6. $(25a^2b^2)^3$.

7. $(-xyz)^{10}$.

8. $(4a^{2x}b^4)^4$.

9. $(3\,mn)^2$.

10. $(\frac{1}{4}\,m^2c)^4$.

11. $\left(\frac{-\,mnp}{3}\right)^4$.

12. $(27\,x^2y^4)^4$.

13. $4(a^2bc)^3$.

14. $(8\,a^3b^3)^5$.

15. $\frac{1}{2}(\frac{1}{2}\,xz^2)^3$.

16. $(6\,a^2b^4)^3$.

17. $-\,4\,a(a^2bc^3)^3$.

18. $(64\,x^3y^6)^3$.

19. $5(-\,2\,x^2y)^4$.

20. $(36\,x^6y^6)^3$.

93. The Square of the Sum of Two Numbers.

Example. $(a+b)^2 = (a+b)(a+b)$, by definition of exponent, § 14.

From the usual work of multiplication of these two binomials which follows

$$\begin{array}{lll} a & +\,b & \\ a & +\,b & \\ \hline a^2 & +\,ab & \\ & +\,ab & +\,b^2 \\ \hline a^2 & +\,2\,ab & +\,b^2, \end{array}$$

we have formula

$$\text{XIII} \qquad \mathbf{(a+b)^2 = a^2 + 2\,ab + b^2.}$$

In like manner, $(3\,a + 2\,b)^2 = 9\,a^2 + 2 \cdot 3\,a \cdot 2\,b + 4\,b^2$
$= 9\,a^2 + 12\,ab + 4\,b^2$.

Check. $(3\,a + 2\,b)^2 = (3+2)^2$, if $a = b = 1$
$= (5)^2 = 25$.

And $9\,a^2 + 12\,ab + 4\,b^2 = 9 \cdot 1 + 12 \cdot 1 \cdot 1 + 4 \cdot 1$
$= 9 + 12 + 4 = 25$.
$25 = 25$.

Expressed in words Formula XIII means that:

The square of the sum of two numbers equals the square of the first, plus twice the product of the first multiplied by the second, plus the square of the second.

EXERCISE

Write results directly from Formula XIII. Check by substituting convenient numerical values:

1. $(2x + y)^2$.

2. $(a + 5)^2$.

3. $(m + 2n)^2$.

4. $(5 + m)^2$.

5. $(1 + a)^2$.

6. $(a^2b + c)^2$.

7. $(4n^3 + b)^2$.

8. $(3x^2 + 2y^2)^2$.

9. $\left(\frac{a}{2} + b\right)^2$.

10. $\left(\frac{x}{3} + \frac{y}{2}\right)^2$.

11. $(ax + by)^2$.

12. $(3abc + 2c)^2$.

13. $(5a + 2b)(5a + 2b)$.

14. $(m + 3n)(m + 3n)$.

15. $[(a + b) + c]^2$.

16. $27^2 = (20 + 7)^2$.

17. $14^2 = (10 + 4)^2$.

18. $(a^p + 2b^q)^2$.

19. $(a^{2c+1} + b^{p+q})^2$.

20. $2(x^p + 3y^{2c})^2$.

94. The Square of the Difference of Two Numbers.

Example. $(a - b)^2 = (a - b)(a - b)$, by definition of exponent, § 14.

From the usual work of multiplying these binomials

$$\begin{array}{llll}
a & -b & & \\
a & -b & & \\
\hline
a^2 & -ab & & \\
 & -ab & +b^2 & \\
\hline
a^2 & -2ab & +b^2, &
\end{array}$$

we have the formula

$$\text{XIV} \qquad (\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 - 2\,\mathbf{ab} + \mathbf{b}^2.$$

This means that:

The square of the difference of two numbers equals the square of the first, minus twice the product of the first multiplied by the second, plus the square of the second.

In like manner, $(3x - 2y)^2 = 9x^2 - 2 \cdot 3x \cdot 2y + 4y^2$
$= 9x^2 - 12xy + 4y^2$.

Check. Let $x = y = 1$.
Then $(3x - 2y)^2 = (3 - 2)^2 = 1^2 = 1$.
And $9x^2 - 12xy + 4y^2 = 9 - 12 + 4 = 1$.
$1 = 1$.

EXERCISE

Write results directly from Formula XIV and check.

1. $(x - 2y)^2$.
2. $(2x - y)^2$.
3. $(n - v)^2$.
4. $(2a - 3b)^2$.
5. $(1 - y)^2$.
6. $(m - 1)^2$.
7. $(a^p - b^q)^2$.
8. $[(x + y) - a]^2$.
9. $[m - (n + p)]^2$.
10. $\dfrac{(x - y)^2}{4}$.
11. $(a - \frac{3}{2})^2$.
12. $(m^2 - n^2)^2$.
13. $\left(\dfrac{x}{2} - y\right)^2$.
14. $(4a^2 - m^2n^2)^2$.
15. $(3x^3 - 2b)(3x^3 - 2b)$.
16. $(\frac{1}{2}p - \frac{1}{3}t)(\frac{1}{2}p - \frac{1}{3}t)$.
17. $(2x^{3c} - 4y^{2p})^2$.
18. $89^2 = (90 - 1)^2$.
19. $29^2 = (30 - 1)^2$.
20. $(896)^2 = (900 - 4)^2$.

95. Factors of Trinomial Squares. From Formula XIII, we have

$$\text{XIII}' \qquad \mathbf{a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b).}$$

Example. $4a^2 + 12ab + 9b^2 = (2a + 3b)^2$
$= (2a + 3b)(2a + 3b)$.

In this example we observe that

(1) $2a = \sqrt{4a^2}$ and $3b = \sqrt{9b^2}$.
(2) $\sqrt{4a^2 + 12ab + 9b^2} = 2a + 3b$.

Again, from Formula XIV we have

$$\text{XIV}' \quad \mathbf{a^2 - 2\,ab + b^2 = (a - b)^2 = (a - b)(a - b).}$$

Example. $9\,a^2 - 24\,ab + 16\,b^2 = (3\,a - 4\,b)^2$
$= (3\,a - 4\,b)(3\,a - 4\,b).$

In this example we observe that:

(1) $3\,a = \sqrt{9\,a^2}$ and $4\,b = \sqrt{16\,b^2}$.

(2) $\sqrt{9\,a^2 - 24\,ab + 16\,b^2} = 3\,a - 4\,b$.

From the foregoing examples it appears that:

(1) *One of the two equal factors of a trinomial square of the form* $a^2 + 2\,ab + b^2$ *is the sum of the square roots of the first and the last terms of the trinomial.*

(2) *One of the two equal factors of a trinomial square of the form* $a^2 - 2\,ab + b^2$ *is the difference of the square roots of the first and the last terms of the trinomial.*

EXERCISE

Factor the following and check results. Point out those that are perfect squares.

1. $x^2 - 6\,x + 9$.
2. $4\,a^2 + 4\,ab + b^2$.
3. $1 - 2\,a^2 + a^4$.
4. $16\,a^2b^2 + 16\,abc + 4\,c^2$.
5. $9\,x^6 - 12\,x^3y^2 + 4\,y^4$.
6. $25\,c^2 + 10\,c + 1$.
7. $64\,x^2 + 80\,xy + 25\,y^2$.
8. $49\,t^2 + 168\,tx + 144\,x^2$.
9. $a^4 - 16\,a^2 + 64$.
10. $16\,u^4 - 72\,u^2v^2 + 81\,v^4$.
11. $121\,p^2 - 264\,pq + 144\,q^2$.
12. $\frac{x^4}{9} + \frac{x^2y^2}{3} + \frac{y^4}{4}$.
13. $1 + \frac{1}{2}g + \frac{g^2}{16}$.
14. $(a - b)^2 - 2\,c(a - b) + c^2$.
15. $4 + 12(m + n) + 9(m + n)^2$.
16. $v^2 - 2\,vt + t^2 + 4(v - t)s + 4\,s^2$.

17. $9e^2 - 24e(T + t) + 16(T + t)^2$.

18. $\frac{m^4y^2}{16} + \frac{3m^2y}{2} + 9$.

19. $\frac{g^2t^2}{4} - 2(A + a)gt + 4(A + a)^2$.

20. $(a - b)^2 - 2(a - b)(c - d) + (c - d)^2$.

21. $529 = (400 + 120 + 9)$. 22. $a^{4n} - 4a^{2n} \cdot b^c + 4b^{2c}$.

23. $a^{2p} + 16a^pb^{2q} + 64b^{4q}$. 24. $4a^{6x} - 12a^{3x}c^y + 9c^{2y}$.

25. $4k^2 + 4k + 1$. 26. $24p^2q^2 + 9p^4 + 16q^4$.

27. $c^4 + 4e^4 + 4c^2e^2$. 28. $169x^2 - 156x^2y^2z^2 + 36y^4z^4$.

29. $121a^2 - 220abc + 100b^2c^2$. 30. $25x^4 + 4y^4 - 20x^2y^2$.

96. The Cube of a Binomial.

Example. $(x + y)^3 = (x + y)(x + y)(x + y)$, by definition of exponent.

Multiplied out in the usual way we find that

XV $$(\mathbf{x} + \mathbf{y})^3 = \mathbf{x}^3 + 3\mathbf{x}^2\mathbf{y} + 3\mathbf{x}\mathbf{y}^2 + \mathbf{y}^3.$$

In like manner

XVI $$(\mathbf{x} - \mathbf{y})^3 = \mathbf{x}^3 - 3\mathbf{x}^2\mathbf{y} + 3\mathbf{x}\mathbf{y}^2 - \mathbf{y}^3.$$

Observe:

(1) x^3 = the cube of the first number.

(2) $3x^2y$ = three times the square of the first multiplied by the second.

(3) $3xy^2$ = three times the first multiplied by the square of the second.

(4) y^3 = the cube of the second.

(5) Signs are all + for cube of the sum of two numbers; and alternate from + to − and − to + for the cube of the difference of two numbers.

Check. For $(x+y)^3$, let $x = y = 1$.

$$\begin{aligned}
\text{Then } (x+y)^3 &= (1+1)^3 = 2^3 = 8,\\
\text{and } x^3 + 3x^2y + 3xy^2 + y^3 &= 1 + 3\cdot 1\cdot 1 + 3\cdot 1\cdot 1 + 1\\
&= 1 + 3 + 3 + 1 = 8.\\
8 &= 8.
\end{aligned}$$

For $(x-y)^3$, let $x = 2$ and $y = 1$.

$$\begin{aligned}
\text{Then } (x-y)^3 &= (2-1)^3 = 1^3 = 1,\\
\text{and } x^3 - 3x^2y + 3xy^2 - y^3 &= 8 - 12 + 6 - 1 = 14 - 13\\
&= 1\\
1 &= 1.
\end{aligned}$$

From the foregoing examples we observe that:

(1) *The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.*

(2) *The cube of the difference of two numbers equals the cube of the first minus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, minus the cube of the second.*

The above result could have been obtained also as follows:

$$\begin{aligned}
(x+y)^3 &= (x+y)^2(x+y).\\
(x+y)^2 &= x^2 + 2xy + y^2 \qquad \text{Formula XIII.}
\end{aligned}$$

$$\begin{array}{rrrrl}
 & & x & +y & \\ \hline
x^3 & +2x^2y & +xy^2 & & \text{product of } x \text{ and } x^2+2xy+y^2.\\
 & x^2y & +2xy^2 & +y^3 & \text{product of } y \text{ and } x^2+2xy+y^2.\\ \hline
x^3 & +3x^2y & +3xy^2 & +y^3 & \text{product of } x+y \text{ and } x^2+2xy+y^2.
\end{array}$$

EXERCISE

Raise each of the following to the power indicated:

1. $(a+b)^3$. 2. $(c-d)^3$.
3. $(m-2n)^3$. 4. $(x-3)^3$.
5. $(1-a)^3$. 6. $(2a-b)^3$.
7. $(3-c)^3$. 8. $\left(\frac{a}{2}-b\right)^3$.
9. $(25)^3=(20+5)^3$. 10. $(75)^3=(80-5)^3$.

97. The Square of a Polynomial.

Example. $(a+b+c+d)^2=[(a+b)+(c+d)]^2$

$$=(a+b)^2+2\cdot(a+b)(c+d)+(c+d)^2$$

$$=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2.$$

Arranging so as to have square terms first,

$$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

Hence, formula,

$$\text{XVII}\quad \mathbf{(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.}$$

From the foregoing example it will be seen that:

The square of the polynomial may be found by grouping the terms so as to change the form of the expression from a polynomial to a binomial and then applying the rule for squaring a binomial.

Taking the terms as arranged above (XVII), it is seen that

a^2 is the square of a,
b^2 is the square of b,
c^2 is the square of c,
d^2 is the square of d.

$2ab$, $2ac$, and $2ad$ are twice the products obtained by multiplying a by each term that follows the first or a term; $2bc$ and $+2bd$ are, likewise, twice the products obtained by multiplying b by each term that follows it; $+2cd$ is twice the product obtained by multiplying c by $+d$, the only term that follows it.

From the above it appears that the same result would have been obtained also by the following rule:

The square of a polynomial equals the sum of the squares of each term of the polynomial, plus twice the products obtained by multiplying each term of the polynomial in succession by all the terms that follow it.

EXERCISE

Perform the operations indicated on each of the following according to rule last given and check in the usual way.

1. $(a - b + c)^2$.
2. $(x - y + s + w)^2$.
3. $(3a + b + 2c)^2$.
4. $(a^2 - b^2 - c^2 - d^2)^2$.
5. $(x^2 - 2xy + y^2)^2$.
6. $(x^3 - x^2y + xy^2 - y^3)^2$.
7. $(m^2 + mn + n^2 + 2)^2$.
8. $(1 - a + b^2 - c^3 - d^4)^2$.
9. $(x^x + b^y + c^p - d^q)^2$.
10. $\left(2a^3 - \frac{c}{3} - \frac{e^2}{2} + 3q\right)^2$.

98. Factors of Polynomials. Three types are to be condered:

TYPE I. Polynomials that are perfect powers.

TYPE II. Polynomials that may be expressed as the difference of the squares of two numbers.

TYPE III. Polynomials that may be factored by grouping terms.

TYPE I. From Formulas XV and XVI, we have

$$\text{XVIII} \quad \mathbf{a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3}$$
$$\mathbf{= (a \pm b)(a \pm b)(a \pm b).}$$

In the foregoing we observe that

(1) One of the three equal binomial factors into which polynomials of this type may be resolved is made up of the algebraic sum of the cube roots of the first and fourth terms of the given polynomial, due attention being given to signs.

(2) The cube root of a polynomial of this type is one of these equal factors.

Thus $\sqrt[3]{a^3 \pm 3a^2b + 3ab^2 \pm b^3} = a \pm b.$

Example 1. $8x^3 - 36x^2y + 54xy^2 - 27y^3 = (2x - 3y)^3$.

For, $8x^3 = (2x)^3$; $-36x^2y = 3(2x)^2(-3y)$; $54xy^2 = 3(2x)(-3y)^2$ and $-27y^3 = (-3y)^3$.

Hence, $8x^3 - 36x^2y + 54xy^2 - 27y^3 = (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$,

which is identical with the form, $(a)^3 - 3(a)^2b + 3(a)(b)^2 - (b)^3$.

Therefore, $2x - 3y$ is the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$, whose factors are $(2x - y)$, $(2x - y)$ and $(2x - y)$.

Again from Formula XVII we have

$$\text{XVII}' \quad \mathbf{a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd} \\ \mathbf{=(a+b+c+d)^2} \\ \mathbf{=(a+b+c+d)(a+b+c+d).}$$

From the foregoing it will be seen that

(1) The two equal factors of any polynomial that can be expressed in the form of the first member of XVII′ are polynomials consisting of the algebraic sum of the square roots of the second degree terms of the given polynomial with due attention to signs.

(2) The square root of a polynomial of this type is one of these equal factors.

Example 2.

$$4x^2 - 12xy + 4xz + z^2 + 9y^2 - 6yz$$
$$= 4x^2 + 9y^2 + z^2 - 12xy + 4xz - 6yz = (2x - 3y + z)^2.$$

For, $(2x)^2 = 4x^2$; $(-3y)^2 = 9y^2$; $(z)^2 = z^2$; $2(2x)(-3y) = -12xy$; $2(2x)(z) = 4xz$ and $2(-3y)(z) = -6yz$.

Hence, $4x^2 + 9y^2 + z^2 - 12xy + 4xz - 6yz = (2x - 3y + z)^2$ and $2x - 3y + z =$ the square root of the given polynomial, whose factors are, therefore, $(2x - 3y + z)$ and $(2x - 3y + z)$.

Type II. **Example 3.**

Factor $a^2 + 2ab + b^2 - c^2$.

$$a^2 + 2ab + b^2 = (a + b)^2, \qquad \text{(XIII}')$$

then $a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2 =$

$$(a + b + c)(a + b - c). \qquad \text{(VIII}')$$

Example 4.

Factor $a^2 + 2ab + b^2 - x^2 + 2xy - y^2$.

$a^2 + 2ab + b^2 = (a + b)^2$, and $-x^2 + 2xy - y^2 = -(x - y)^2$, (XIV′)

then $a^2 + 2ab + b^2 - x^2 + 2xy - y^2 = (a + b)^2 - (x - y)^2$

$$= (a + b + x - y)(a + b - x + y). \qquad \text{(VIII}')$$

Example 5. Factor $a^4 + a^2b^2 + b^4$.

$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2$, by adding and subtracting a^2b^2,

$= (a^2 + b^2)^2 - a^2b^2$, as in Example 3,

$= (a^2 + b^2 + ab)(a^2 + b^2 - ab)$. (VIII′)

Type III. It frequently happens that a factor is common to all the terms of an expression. In such cases the common factor should be removed first. Very often by grouping terms a common factor to all the groups will appear.

Example 6. Factor $mx^2 + nx + px^3 + qx$.

In this example x is seen to be common to all terms. Removing x we have $mx^2 + nx + px^3 + qx = x(mx + n + px^2 + q)$. As nothing further can be done to factor the quantity in parentheses, (x) and $(mx + m + px^2 + q)$ are the factors of the given polynominal.

Example 7. Factor $mx + nx - my - ny$.

$$\begin{aligned} mx + nx - my - ny &= (mx + nx) - (my + ny) \\ &= x(m + n) - y(m + n) \\ &= (m + n)(x - y), \text{ } (m + n) \text{ being common to both terms.} \end{aligned}$$

EXERCISE

Factor:

1. $x^3 + 3x^2 + 3x + 1$. **2.** $x^2 - xy - xz + yz$.

3. $4a^4 + 3a^2b^2 + b^4$. **4.** $m^2 + m^3 - m - 1$.

5. $4a^2 - 8ab + 4b^2 - c^2$. **6.** $9x^3 + 18x^2 + 9x$. (First remove common factor.)

7. $a^2 + 2a + 1 - 4c^2 + 4cd - d^2$.

8. $m^3 - 9m^2p + 27mp^2 - 27p^3$.

9. $s^3 - s^2t + rs - rt$. **10.** $8 + 12z + 6z^2 + z^3$.

11. $1 + t^2 + v^2 + 2t - 2u + 2v - 2tu - 2uv + u^2$.

12. $a^2 - b^2 + a + b$. **13.** $a^3 + a + b^3 + b$.

14. $z^3 + 2z^2 + 2z + 1$.

15. $27t^3 - 108t^2v^4 + 144tv^8 - 64t^{12}$.

16. $3a^4 + 6a - a^3 - 2$. **17.** $4m^2 - 12mn - p^2 - 9n^2$.

18. $9y^2 - m^2 - 2mn - n^2$. **19.** $4b^2 + 49a^2 - 28ab - 16c^2$.

20. $c^4 + c^5 + 4 + 4c$. **21.** $6m^2 - 4mn + 9mp - 6np$.

22. $3a^3 - 7 - 7a^2 - 3a$. **23.** $3 + 6xy - 3y^2 - 3x^2$.

24. $a^2 - 4a + 4 - 4b^2$.

25. $100y^4 - 4a^4 - 24a^2b^2 - 36b^4$.

26. $x^4 + 3x^3y + 3x^2y^2 + xy^3$.

27. $-10c^2mn + 5c^2n^2 - 20c^2p^2 + 5c^2m^2$.

28. $rs - 3x + zy - sz - ry + rx$.

29. $(a - b) - (a - b)(c - d)^2$. **30.** $9m^4 - 7m^2n^2 + n^4$.

31. $k^2 - 2gk + g^2 + 1 + 2k - 2g$.

32. $v^2t^2 - \frac{1}{2}g^2t^2 + \frac{1}{16}g^4t^2 - t^4$. **33.** $c^2 + 2cd + d^2 + 5c + 5d$.

34. $3y^4 - 3y^3 - 720y^2$. **35.** $8a^4 - 65a^3 + 8a^2$.

36. $x^2 - 2xy + y^2 - (x - y)(x + y)$.

37. $a^4 + a^3 + a^2 - 1$.

38. $a^3 + b^3 + a^2 - b^2$. **39.** $b^3 - b^2 - 4b + 4$.

40. $y^{4a} - 13y^{2a} + 36$. (Factor in two ways.)

(b) Quadratic Equations

99. The Quadratic Trinomial, $ax^2 + bx + c$, which has been factored in § 89, when equated to zero, gives

$$ax^2 + bx + c = 0,$$

as the most general form of the quadratic equation. Since many quadratic equations, when put into the above form, are readily factored and a solution thereby obtained, we shall begin the study of these equations at this point.

100. Definitions. The degree of an equation in one unknown number is determined by the highest power of the unknown number that occurs in the equation (see § 49). An equation that contains no higher power of the unknown than the first is an equation of the first degree and is called linear. An equation that contains the second power but no higher power than the second of the unknown is of the second degree and is called quadratic.

For example:

$3x + 4 = 7$ is of the first degree in x and is called linear.

$x^2 + 3x - 10 = 0$ is of the second degree in x and is called quadratic.

A *root of an equation,* as defined in § 53 *e,* is a number that satisfies the equation; that is, if it is substituted for the letter that represents the unknown number the equation becomes an identity. Usually an equation has the same number of roots as the number that designates its degree. Since some of the roots may be equal there may not be as many separate and distinct roots as the number designating the degree of the equation; but in no case can there be a greater number of roots than this number that denotes the degree of the equation. Hence, a quadratic equation usually has two separate roots but no more than two.

101. Solution of the Quadratic by Factoring.* Whenever the quadratic may be factored its solution may be completely effected; that is, its two roots may be found, by equating each of these factors to zero. This solution of the quadratic is based upon the principle established in (§ 67), which may be stated as follows:

If a product of two or more factors is zero, the statement will be true if any factor equals zero, for zero times any finite number is zero. Hence, any factor equated to zero may contain a root, since a root is nothing more or less than a value for the unknown which satisfies the equation.

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* The solution of the general equation of the nth degree by factoring is based upon the factor theorem which will be stated and illustrated in Chap. XII.

Example.

Solve $x^2 + 3x + 2 = 0$.

Solution.

$(x + 2)(x + 1) = 0$, factoring given equation.

First, suppose that (1) $x + 2 = 0$, we would have $0 \times (x + 1) = 0$, which is true.

Hence, $x = -2$, obtained from (1) by subtracting 2 from both sides, is the value of x, which makes $x + 2 = 0$.

Substituting -2 for x in the given equation we have

$$\begin{aligned} (-2)^2 + 3(-2) + 2 &= 0 \\ 4 - 6 + 2 &= 0 \\ 6 - 6 &= 0 \\ 0 &= 0. \end{aligned}$$

Hence it appears that -2 is a root of the equation.

Next, suppose that (2), $x + 1 = 0$; we would have $(x + 2) \times 0 = 0$, which is also true.

Hence, $x = -1$, obtained from (2) by subtracting 1 from both sides, is the value of x, which makes $x + 1 = 0$.

Substituting -1 for x in the given equation gives

$$\begin{aligned} (-1)^2 + 3(-1) + 2 &= 0 \\ 1 - 3 + 2 &= 0 \\ -3 + 3 &= 0 \\ 0 &= 0. \end{aligned}$$

And hence it is seen that -1 is also a root of the given equation.

Therefore, the two roots of $x^2 + 3x + 2 = 0$ are -2 and -1.

From the foregoing example it will be observed that the roots of a quadratic are obtained as follows:

(1) *Put the equation in the form* $ax^2 + bx + c = 0$, *if not already in this form.*

(2) *Factor the first member of this equation.*

(3) *Put each of the factors separately equal to zero, thus forming two linear equations of the form* $ax + c = 0$.

(4) *Solve these linear equations in the usual manner.*

(5) *Check results to see if they satisfy the equation.*

EXERCISE

102. Solve each of the following equations by factoring, as in § 101, and check answers:

1. $x^2 - x - 6 = 0$.
2. $x^2 - 5x = -6$.
3. $x^2 + 3x = 10$.
4. $2x^2 - 7x + 3 = 0$.
5. $6x^2 - 13x + 6 = 0$.
6. $12x^2 - 3x - 9 = 0$.
7. $7x^2 + x = 21 - 3x^2$
8. $6s^2 - 17s + 12 = 0$.
9. $6y^2 + y = 12$.
10. $6y^2 + 7y = 3$.
11. $3y^2 + 7y = 6$.
12. $6y(y - 3) = 5(y - 3)$.
13. $(z-1)(z^2-12z+27)=0$.
14. $4y^3 + 17y^2 + 4y = 0$.
15. $y^3 - 11y^2 + 18y = 0$.
16. $60t^2 - 59t = 20$.
17. $y(y - 6) = 7y - 42$.
18. $4n^2 - n - 14 = 0$.
19. $5z^2 + 26z = -5$.
20. $2e^2 + 2e = 40$.
21. $12L^2 - 23L + 10 = 0$.
22. $8w^2 + 8w = 6$.
23. $8v^2 + 10v - 12 = 0$.
24. $8n^2 + 22n + 15 = 0$.
25. $7r^2 + r - 6 = 0$.
26. $4y^2 - 8y + 4 = 0$.
27. $12x^2 - 9x - 30 = 0$.
28. $p^2 - 2p - 3 = 0$.

PROBLEMS LEADING TO QUADRATICS

(Solve by Factoring).

103. **1.** Find two numbers whose difference is 5 and whose product is 36.

2. A room is four feet longer than it is wide, and its area is 96 square feet. What are its dimensions?

3. The depth of a city lot which has an area of 3750 square feet is six times its frontage. Find its frontage.

4. The length of a box is three times its depth and its width is two inches greater than its length. The area of its entire outer surface is 88 square inches. Find its dimensions.

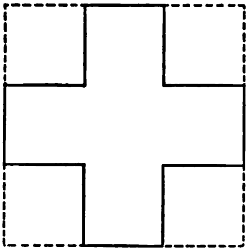

FIG. 30

5. Find cubic contents of the box in Example 4.

6. If the speed of a train were five miles an hour greater, it would require one hour less time to travel 150 miles. What is the rate of the train?

7. A company of tourists engaged a guide for $80 and agreed to share the expenses equally. Four of the company, however, withdrew and it was found that each of the others would have to pay one dollar more than he had expected to pay. How many tourists were in the original party?

8. An open box 8 inches deep is to be cut out of a square piece of tin by cutting out an 8-inch square piece from each corner and folding up the sides. If the box is to hold 3200 cubic inches, what must be the length of a side of the square piece of tin? (Fig. 30.)

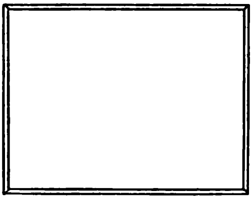

FIG. 31

9. A picture 16 inches by 20 inches is enclosed in a frame of such width that the area of the frame is 140 square inches greater than the area of the picture itself. What is the width of the frame? (Fig. 31.)

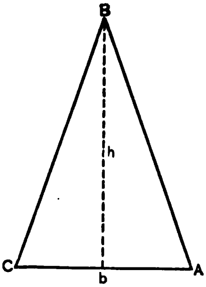

FIG. 32

The *altitude* of a triangle is the perpendicular distance from any vertex to the opposite side. This side is called the *base.*

The *area* of any triangle is one-half the product of its base and altitude. In Fig. 32, h denotes altitude and b denotes base. The expression $\frac{hb}{2}$ denotes the area.

10. The altitude of a triangle is 2 inches greater than its base and the area of the triangle is 60 square inches. Find the base.

11. The area of a triangle is 48 square feet and its base is four feet longer than its altitude. Find the altitude.

12. The end of a barn, as shown in Fig. 33, consists of two parts: The gable, triangular part; and the body, rectangular part. The altitude of the triangle, or the height of gable, is one-third of the width of the barn and the width of the barn is ten feet greater than the height of the body or rectangular part. If the total area of one end of the barn is three times the area of the gable, find the width of the barn.

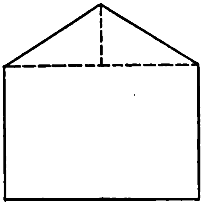

FIG. 33

MISCELLANEOUS EXAMPLES IN FACTORING FOR REVIEW

104. Factor:

1. $4x^4 + 8x^2y^2 + 9y^4$.

2. $4x^4 - 4x^2y^2 + 9y^4$.

3. $5y^2 + 5 + 3y + 3y^3$.

4. $y^2z^2 - 4z^2 + 4 - y^2$.

5. $m^4 + n^4 - 16 + 2m^2n^2$.

6. $1 + 4m - 4n^2 + 4m^2$.

7. $az^2 + ay^2 + by^2 + bz^2$.

8. $r^4 + 64 - 16r^2 - m^2$.

9. $cy + 2cx + 2ay + 4ax$.

10. $(5x^2 - 26y^2)^2 - (4x^2 - 10y^2)^2$.

11. $a^3 - a^2 - a + 1$.

12. $5x^2 - 8xy + 3y^2$.

13. $3x^3 - x^2 - 12x + 4$.

14. $y^2 - a^2 - y - a$.

15. $(x + y + xy)^2 - (x - y)^2$.

16. $(a^3 - 1) + (3a^2 - a - 2)$.

17. $(x + 2y)^2 - 9y^2$.

18. $(a + b - c)^2 - (a - c)^2$.

19. $a^2b^2 - a^2 - b^2 + 1$.

20. $(a + 2b + 3c)^2 - 4(a + b - c)^2$.

21. $a^2(a + b - c)^2 - c^2(b + c - a)^2$.

22. $z^4 - z^2 - z + 1$.

23. $z^5 - z^3 + z - 1$.

24. $x^4 + yx^3 + z^3x + z^3y$.

25. $r^2s^2 - r^2 - s^2 + 1$.

26. $(13a^2 - 5b^2)^2 - (12a^2 + 4b^2)^2$.

27. $a^2(a+b)^2-(a^2+b^2)^2$. **28.** $y^4+4y^2+4-y^6$.
29. $6cd-15ce+22rd-55re$. **30.** $36x^2-29xy+5y^2$.
31. $26+39n-22m-33mn$. **32.** $ac+ad^3-b^4c-b^4d^3$.
33. $a^2+4ab+4b^2-(a^2-4ab+4b^2)$.
34. $(3z-1)^2-(x^2+4y^2-4xy)^2$.
35. $(x+3y)^2+(x-2y)^2+2(x+3y)(x-2y)$.
36. $16(m+n)^2-8(m+n)(m-n)+(m-n)^2$.
37. $256z^2-(49z^2+4t^4-28zt^2)$.
38. $(2s-a)^2+100(a-3s)^2+20(2s-a)(a-3s)$.
39. $-48(c-d)(c+d)+36(c-d)^2+16(c+d)^2$.
40. $(a^2+x^2)^2-(a+x^2)^2$.
41. y^a+y^{2a}. (Suggestion: $y^{2a}=y^a\cdot y^a$. Also $y^{b+q}=y^b\cdot y^q$ and $c^{x+1}=c^x\cdot c^1$.)
42. y^n+y^{a+n}. **43.** $4a^3+4a-4a^2b-4b$.
44. y^4+y^2+1. **45.** $1+x-x^2yz-x^3yz$.
46. b^2-b^3+1-b. **47.** $x^4-7x^2y^2+y^4$.
48. $r^{2a}-19r^a+48$. **49.** $49r^4s^4-53r^2s^2t^2+4t^4$.
50. $16c^4-17c^2d^2+d^4$. **51.** $x^ay^{2b}+x^{2a}y^b$.
52. $k^{a+5}-k^{a+3}-k^{a+1}+k^{a-1}$. **53.** $a^{10x}+23a^{5x}b^{3y}+102b^{6y}$.
54. $c^2d^8+c^2d^4+c^2$. **55.** $a(p+q)^3+p+q$.
56. $25a^{7x}b^{5y}+15a^{3x}b^{5y}$. **57.** $b^{p+q}+b^qc^r+b^pc^q+c^{q+r}$.
58. $25x^4+31x^2y^2+16z^4$. **59.** $3s^{r+2}-5s^rt^2+3s^2t^u-5t^{u+2}$.
60. $s^{6t}-7s^{3t}-78$.
61. $b^{2n}-6b^nc^p+9c^{2p}+7b^nd-21c^pd-44d^2$.
62. $x^{2a}y^{3b}-3x^{3a}y^{2b}+2x^{4a}y^b$.
63. $169(x+y)^2-78(x+y)(p+q)^3+9(p+q)^6$.
64. $(a+b)^{2n}-7(a+b)^n+10$.
65. $a^4+2a^2b^2+b^4+3a^2+3b^2-10$.
66. $k^{x+y}\cdot p^{a+b}+k^{y+z}\cdot p^{a+c}$. **67.** $x^{2m}-20x^my^{p^3}-69y^{2p^3}$.
68. $6x^{2a+b}\cdot y^{3c+2d}-18x^{a+b}\cdot y^{c+2d}$.
69. $(p+q)(r+s)-2s(p+q)$.
70. $2c^{x+1}-3cd^y-2c^xd+3d^{y+1}$.

Give orally the products to the following:

71. 19×21.

72. $(39)^2$.

73. 38×42.

74. $(48)^2$.

75. 47×53.

76. $(61)^2$.

77. 36×44.

78. $(17)^2$.

79. 25×35.

80. $(86)^2$.

CHAPTER VII [*]

SQUARE ROOT—RADICALS

(a) Square Root

105. Square Root of a Polynomial. Since

$$[\pm (x + y)]^2 = x^2 + 2xy + y^2,$$

it follows that the square root of

$$x^2 + 2xy + y^2 = \pm (x + y).$$

The method of finding the square root of a polynomial will now be illustrated, using $x^2 + 2xy + y^2$ as the given polynomial whose square root is desired.

$$
\begin{array}{rl|l}
 & \textbf{Polynomial} & \textbf{Root} \\
 & x^2 + 2xy + y^2 & \underline{x + y} \\
(x)^2 = & \underline{x^2} & \\
\text{Trial divisor } = 2x & \multicolumn{1}{|l}{2xy + y^2} & \\
\text{Comp. div. } = 2x + y & \multicolumn{1}{|l}{\underline{2xy + y^2 = (2x + y)(y)}} &
\end{array}
$$

Explanation. Since $x = \sqrt{x^2}$, x is the first term of the root. $2x =$ the trial divisor, since the trial divisor is twice that part of the root already found.

---

[*] Chapter IX, which treats of algebraic fractions, may be taken before Chapter VII, by those who desire to have a simple treatment of fractions follow immediately after factoring.

$2xy \div 2x = y$, hence y is the second term of the root.

y added to $2x$ gives gives $2x + y$, the complete divisor, which multiplied by y gives $2xy + y^2$, the other two terms of the given expression.

Therefore, the required roots are $\pm (x + y)$.

We shall now extend this process and find the square root of a polynomial when the root contains three terms.

Example. Find the square root of $4x^4 - 12x^3 + 13x^2 - 6x + 1$.

Solution.

$$
\begin{array}{lll|l}
 & & \textbf{Polynomial} & \textbf{Root} \\
 & & 4x^4-12x^3+13x^2-6x+1 & 2x^2-3x+1 \\
 & (2x^2)^2 = 4x^4 & & \\
\hline
\text{Trial divisor} & =2\cdot 2x^2=4x^2 & -12x^3+13x^2-6x+1 & \\
\text{Comp. divisor} & =4x^2-3x & -12x^3+9x^2=(4x^2-3x)(-3x) & \\
\hline
\text{2d. Trial div.} & =2(2x^2-3x) & 4x^2-6x+1 & \\
 & =4x^2-6x & & \\
\text{2d Comp. div.} & =4x^2-6x+1 & 4x^2-6x+1=(4x^2-6x+1)(1) & \\
\hline
\end{array}
$$

Therefore, the required roots are $\pm (2x^2 - 3x + 1)$.

Check. $[\pm (2x^2 - 3x + 1)]^2 = 4x^4 - 12x^3 + 13x^2 - 6x + 1$.

Hence the result is correct.

From the foregoing solutions it will be seen that:

(1) *The terms of the polynomial are arranged according to the powers of some letter.*

(2) *The first term in the root is the square root of the first term of the given polynomial.*

(3) *The trial divisor is always twice that part of the root already found.*

(4) *Each term of the root after the first is found by dividing the first term of the remainder, obtained by subtracting the product of the complete divisor and the preceding term of the root from the polynomial, by the trial divisor.*

(5) *Since the square root of a positive number is both positive and negative, the signs + and − should be written before the root as a whole by enclosing the root in ( ) preceded by ±.*

EXERCISE

Write out from inspection the square roots of all the trinomials in the following exercise; determine the square roots of the other polynomials by the method illustrated above:

1. $x^4 - 4x^2y + 4y^2$.

2. $4 + 8c + 4c^2$.

3. $1 - 6y + 9y^2$.

4. $\frac{1}{e^2} - 2 + e^2$.

5. $\frac{a^2}{b^2} + 2 + \frac{b^2}{a^2}$.

6. $25a^2b^2 + 60abc + 36c^2$.

7. $x^2 + y^2 + 2xy + z^2 + 2xz + 2yz$.

8. $y^6 + 4ay^5 + 10a^3y^3 + 4a^5y + a^6$.

9. $1 - 4c + 10c^2 - 20c^3 + 25c^4 - 24c^5 + 16c^6$.

10. $1 - 2x + 2x^2 - x^3 + \frac{x^4}{4}$.

11. $49 - 42t + 37t^2 - 12t^3 + 4t^4$.

12. $4b^2c^2 - 16abc^2 + 16a^2c^2 - 12ab^2c + 24a^2bc + 9a^2b^2$.

13. $a^2 + 2ab - 3b^2 - \frac{4b^3}{a} + \frac{4b^4}{a^2}$.

14. $9e^{10} + 6e^8 + e^6 - 6e^5 - 2e^3 + 1$.

15. $36x^6 + 12x^4 - 60x^3 + x^2 - 10x + 25$.

16. $\frac{m^2}{n^2} + \frac{n^2}{m^2} + \frac{17}{4} - \frac{5m}{n} + \frac{5n}{m}$.

17. $\frac{4x^2}{y^2} + 7 + \frac{y^2}{4x^2} + \frac{12x}{y} - \frac{3y}{x}$.

18. $\frac{9}{a^2} + \frac{a^2}{4} - 3.$

19. $1 + \frac{4}{a} + \frac{10}{a^2} + \frac{20}{a^3} + \frac{25}{a^4} + \frac{24}{a^5} + \frac{16}{a^6}.$

20. $\frac{25}{x^2} + \frac{30}{x} + 69 + 36x + 36x^2.$

21. $9y^4 - 2y^3 - \frac{161y^2}{9} + 2y + 9.$

22. $a^4 - 2a^3 + a + \frac{1}{4}.$

23. $b^2 - 2 + \frac{1}{b^2}.$

24. $\frac{1}{9} - \frac{a}{3} + \frac{9a^4}{16} - \frac{3a^3}{4} + \frac{3a^2}{4}.$

25. $n^4 - 2n^3 + 2n^2 - n + \frac{1}{4}.$

106. The Square Root of Arithmetical Numbers. Beginning at units and pointing off the number into periods of two figures each will show how many figures the root contains, since to each figure in the root there will be one period in the number. For, squaring a number consisting of one digit will give a number consisting of not more than two digits; example, $9^2 = 81$. The square of a number consisting of two digits will give a number consisting of not more than four digits; example, $99^2 = 9801$. The square of a number containing three digits consists of not more than six digits, etc. Hence, in extracting square root, pointing the number off into periods of two figures each will indicate the number of figures to be found in the root.

After pointing off the number into periods of two figures each, the remainder of the process is the same as that for finding the square root of an algebraic polynomial, as we shall now show.

For a number consisting of two digits, let t stand for the digits in tens place, and u the digit in units place. Then the square root of $t^2 + 2tu + u^2$, found as in § 105 is $\pm (t + u)$, the work being given below:

$$
\begin{array}{rl|l}
 & t^2 + 2tu + u^2 & \underline{t+u} \\
\text{Square of } t \quad = & \underline{t^2} & \\
\text{Trial divisor } = 2t & \multicolumn{2}{|l}{2tu + u^2} \\
\text{Comp. divisor} = 2t + u & \multicolumn{2}{|l}{\underline{2tu + u^2 = (2t + u)(u)}}
\end{array}
$$

Therefore, $\pm (t+u)$ are the square roots of $t^2+2tu+u^2$.

Applying the foregoing process to find the square root of 1225, gives

$$
\begin{array}{rl|l}
\pm (t + u)^2 = & t^2 + 2tu + u^2 & t + u \\
1225 = & 900 + 300 + 25 & \underline{30 + 5 = 35} \\
t = \text{3-tens} = 30;\ t^2 = (30)^2 = & \underline{900} & \\
\text{Trial divisor} = 2t = 2 \times 30 = 60 & \multicolumn{2}{|l}{300 + 25} \\
2tu \div 2t = u & \multicolumn{2}{|l}{} \\
300 \div 60 = 5 & \multicolumn{2}{|l}{} \\
\text{Complete divisor} = 2t + u & \multicolumn{2}{|l}{} \\
= 60 + 5 & \multicolumn{2}{|l}{\underline{300 + 25 = (60 + 5)(5)}}
\end{array}
$$

Therefore, ± 35 are the square roots of 1225.

In practical work, the above is abbreviated to the following:

$$
\begin{array}{r|l}
 & 12'\;25 \quad \big|\, \underline{30 + 5 = 35} \\
t^2 = & \underline{9\;\;00} \\
2t = 2 \cdot 30 = 60 & 3\;\;25 \\
2t + u = 60 + 5 = 65 & \underline{3\;\;25 = (65) \cdot (5)}
\end{array}
$$

Again applying this process to find the square root of 15,625, a number whose square root will consist of three digits, we have

$$
\begin{array}{l|l|l}
 & 1'\,56'\,25 & \underline{100+20+5=125} \\
t^2 = (10\text{-tens})^2 = (100)^2 = & \underline{1\ 00\ 00} & \\
\text{Trial divisor} = 2\,t = 2 \cdot 100 = 200 & 56\ 25 & \\
\text{Complete divisor} & & \\
\quad = 2\,t + u = 200 + 20 = 220 & \underline{44\ 00 = (220) \times (20)} & \\
\text{Trial divisor} & & \\
\quad = 2\,t = 2(100+20) = \quad 240 & 12\ 25 & \\
\text{Complete divisor} & & \\
\quad = 2\,t + u = 240 + 5 = \quad 245 & \underline{12\ 25 = (245) \cdot (5)} &
\end{array}
$$

Therefore ±125 are the square roots of 15,625.

From the foregoing examples we have the following rule for finding the square roots of arithmetical numbers:

(1) *Beginning at the decimal point, point the number off into periods of two figures each going toward the left for the integral part, toward the right for the decimal part; and both ways for mixed numbers.*

(2) *For the first digit in the root find by trial or inspection the greatest number whose square is less than or equal to the first or left-hand period. Square this digit thus determined and subtract its square from the first period.*

(3) *Bring down the second period, annexing it to the remainder obtained by subtracting in* (2).

(4) *Double the part of the root already found for a trial divisor and determine the next figure of the root by dividing the remainder by the trial divisor, always omitting the right-hand figure of the remainder.*

(5) *Complete the divisor by annexing the digit just determined in* (4) *to the trial divisor and multiply the completed*

divisor by this same digit. After subtracting the product from the dividend bring down the next period for a new dividend.

(6) *Double that part of the root thus far determined for a new trial divisor and proceed as in* (4) *and* (5). *Continue in this manner until all the periods have been brought down and used.*

(7) *If the number whose root is being found contains a decimal, point off as many digits in the root, counting from the right, as there are periods of decimals in the given number.*

Check. If the root is exact, square it and see if this gives the original number. If the root is not exact, square and add the remainder. This sum should be the original number.

EXERCISE

107. Use the method illustrated in § 106 to find the square roots of the following:

1. 961. **2.** 2401. **3.** 576.

4. .225. (Suggestion. Annex a zero so as to make the period at the left a full period. Every period of a decimal number must contain two figures before beginning the process.)

5. .0441. **6.** .0001. **7.** 15.625.

8. .001. **9.** $\frac{5}{8}$. (First write as a decimal).

10. 3. (Give result to 4 decimal places.)

11. Evaluate $\sqrt{2} \div \sqrt{3}$, giving result to 3 decimal places.

12. Evaluate $(\sqrt{3} - \sqrt{5})(\sqrt{3} + 2\sqrt{5})$, giving result to 3 decimal places.

In each of the following express results as decimals to 3 places:

13. $(\sqrt{8} - 2\sqrt{3})(\sqrt{8} + 2\sqrt{3})$.

14. $(5\sqrt{2} - 4\sqrt{7}) \div \sqrt{2}$. **15.** $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{3}}$.

(b) Radicals

108. The Numbers of Algebra are classed as *real* and *imaginary*.* Such numbers as 2, 3, $\frac{1}{2}$, $\sqrt{3}$, -7, and $\sqrt[3]{5}$ are real numbers. Numbers expressed thus, $\sqrt{-2}$ and $3\sqrt{-5}$ are called imaginary numbers. Numbers like $\sqrt{-2}$ arise when we attempt to express an *even root* of a *negative* number. They frequently occur as roots of equations.

Real numbers are either *rational* or *irrational*. A *rational* number is one that can be obtained from unity by performing the operations of addition, subtraction, multiplication and division. Any integer, positive or negative is rational; also any number that represents the quotient of two integers. Thus $\frac{2}{3}$ is a rational number, since it expresses the quotient of 2 divided by 3 and both 2 and 3 may be obtained from unity by the addition of unity one or more times. Likewise, 7, .25, and $1\frac{1}{3}$ are rational numbers. On the other hand, such numbers as $\sqrt{2}$ and $\sqrt[3]{5}$ are not rational, since they cannot be expressed as the quotient of two integers, nor can they be derived from unity by performing the operations of addition, subtraction, multiplication, and division. All real numbers not rational are irrational.

A *radical* is an indicated root of a number. Thus, $\sqrt{4}$ and $\sqrt{5}$ are radicals; also $\sqrt{b}$ and $\sqrt{x^2 + y^2}$ are radicals. If the number whose root is to be taken is not an exact power of the degree indicated, the radical is called a *surd*. Thus, $\sqrt{5}$ and $\sqrt{10}$ are surds. It will be observed that surds are always irrational.

---

* In the opening paragraph of this book the student was told that numbers unknown to arithmetic would be dealt with in algebra. Negative numbers and imaginaries are two such kinds of numbers that lie beyond the scope of arithmetic.

The radical sign, $\sqrt{}$, as previously explained, indicates that a root is to be found. The sign itself indicates square root. When any other root is to be taken that fact is shown by writing the number (called the Index) indicating the degree of the root, in the $\surd$ part of the radical sign. Thus, $\sqrt[3]{a}$ means that the cube root of a is desired; 3 is the index.

109. Fractional Exponents. In studying the principles of multiplication, § 67, we learned that

$$a^p \cdot a^q = a^{p+q},$$

p and q being integral numbers.

This same law also holds for fractional values of p and q. Hence, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^1 = a$. Then since $a^{\frac{1}{2}}$ multiplied by itself produces a, it follows that the square root of $a = (a)^{\frac{1}{2}} = \sqrt{a}$.

In like manner it may be shown that the cube root of $a = (a)^{\frac{1}{3}} = \sqrt[3]{a}$.

And, in general,

$$\text{the } n\text{th root of } a = (a)^{\frac{1}{n}} = \sqrt[n]{a},\ n \text{ being integral.}$$

In words this means that the *denominator of a fractional exponent indicates a root.*

Again, since $a^{\frac{2}{3}} = a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2,$

and since $a^{\frac{2}{3}} = a^{2 \cdot \frac{1}{3}} = (a^2)^{\frac{1}{3}} = \sqrt[3]{a^2},$

it follows that $(\sqrt[3]{a})^2 = \sqrt[3]{a^2}.$

In particular $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = (+2)^2 = 4,$

and again, $8^{\frac{2}{3}} = 8^{2 \cdot \frac{1}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4.$

Hence, $(\sqrt[3]{8})^2 = \sqrt[3]{8^2},$

and, in general,

XXI $$a^{\frac{p}{s}} = (\sqrt[s]{a})^p = \sqrt[s]{a^p}.$$

This means that *the denominator of a fractional exponent indicates a root and the numerator a power.* Also, that a *power of a root equals the root of the power.*

Because of the relation shown in Formula XXI radical numbers may be expressed either with the radical signs or with fractional exponents.

EXERCISE

110. Write with radical signs:

1. $x^{\frac{3}{5}}$.
2. $(x^2)^{\frac{1}{3}}$.
3. $(x^3)^{\frac{1}{4}}$.
4. $y^{\frac{5}{3}}$.
5. $8^{\frac{3}{2}}$.
6. $(a+b)^{\frac{5}{7}}$.
7. $5(x-y)^{\frac{2}{3}}$.
8. $10 \cdot x^{\frac{2}{5}} + 5\,y^{\frac{3}{4}}$.
9. $4^{\frac{1}{3}} \cdot (x^a - y^b)^{\frac{2}{3}}$.
10. $9^{\frac{3}{2}} \cdot a^{\frac{1}{2}} \cdot b^{\frac{5}{2}}$.

Find value of

11. $4^{\frac{3}{2}}$.
12. $9^{\frac{5}{2}}$.
13. $-27^{\frac{2}{3}}$
14. $-64^{\frac{2}{3}}$.
15. $8^{\frac{5}{3}}$.
16. $(\frac{1}{9})^{\frac{3}{2}}$.
17. $\left(\frac{8}{27}\right)^{\frac{2}{3}}$.
18. $(-0.125)^{\frac{4}{3}}$.
19. $(4\,x^6)^{\frac{3}{2}}$, if $x = 2$.
20. $27\,a^{\frac{p}{q}}$, if $a = 8$, $p = 4$, and $q = 3$.

Write the following with fractional exponents:

21. $2\sqrt[3]{81}$.
22. $\sqrt[3]{56\,x^5}$.
23. $2\sqrt{25\,b}$.
24. $\sqrt{\frac{2}{3}}$.
25. $\sqrt[3]{(300\,a)^4}$.
26. $\sqrt[5]{\left(\frac{3\,a}{2\,b}\right)^2}$.
27. $\sqrt[4]{25\,x^2y^2}$.
28. $\sqrt[3]{250}$.

29. $\sqrt[5]{243\,a^5x^{10}}$. **30.** $\sqrt[3]{bx^4}$.

31. $\sqrt[5]{a^nb^p}$. **32.** $\sqrt[n]{a^5b^6}$.

33. $\sqrt[p]{x^4y^5}$. **34.** $\sqrt[p]{a^q\cdot b^t}$.

35. $\sqrt[p]{a^{2p}b^p}$. **36.** $\sqrt[r]{x^ay^{br}}$.

37. $\sqrt[q]{x^2y^3}$. **38.** $\sqrt[n]{a^{nm}b^{2nm}}$.

39. $\sqrt[n]{(a+b)^m}$. **40.** $\sqrt[n]{(a+b)^p\cdot(a-b)^q}$.

111. Zero and Negative Exponents.

Since $a^p \div a^q = a^{p-q}$,

it follows that $a^3 \div a^3 = a^{3-3} = a^0$.

But $a^3 \div a^3 = 1$ (since $1 \times a^3 = a^3$).

Therefore,

XXII $\mathbf{a^0 = 1}$. (Axiom 5: Things equal to same thing . . .)

Again, since $a^p \div a^q = a^{p-q}$,

it follows that $a^3 \div a^5 = a^{3-5} = a^{-2}$.

$$\text{But}\quad a^3 \div a^5 = \frac{a^3}{a^5} = \frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a} = \frac{1}{a\cdot a} = \frac{1}{a^2}.$$

Therefore,

XXIII $\mathbf{a^{-2} = \frac{1}{a^2}}$. (Axiom 5.)

From Formula XXIII it appears that a number with a negative exponent means the same thing as the reciprocal* of that number with a positive exponent.

$$\text{Thus,}\quad 2\,a^{-3} = 2\cdot a^{-3} = 2\cdot\frac{1}{a^3} = \frac{2}{a^3},$$

$$\text{and}\quad ab^{-n} = a\cdot b^{-n} = a\cdot\frac{1}{b^n} = \frac{a}{b^n}.$$

---

* The reciprocal of a number is unity divided by the number.

Also,
$$\frac{a^{-p}}{b^{-q}} = a^{-p} \div b^{-q} = \frac{1}{a^p} \div \frac{1}{b^q}$$
$$= \frac{1}{a^p} \times \frac{b^q}{1}$$
$$= \frac{b^q}{a^p}.$$

In general,
$$\frac{ab^{-p}}{cd^{-q}} = ab^{-p} \div cd^{-q}$$
$$= a \cdot \frac{1}{b^p} \div c \cdot \frac{1}{d^q}$$
$$= \frac{a}{b^p} \div \frac{c}{d^q}$$
$$= \frac{a}{b^p} \times \frac{d^q}{c}$$
$$= \frac{ad^q}{cb^p}.$$

From which it follows that any number with a negative exponent may be transferred from either term of a fraction to the other, provided such number is a factor of the term in which it stands, by changing the sign of its exponent.

112. Simplification of a Radical consists in changing its form, without changing its value, so that the number under the radical sign

(1) Shall be an integer.
(2) Shall contain no factor that is an exact power corresponding to the degree of the radical.
(3) Shall not be an exact power corresponding to some factor of the number indicating the degree of the radical.

The following are illustrations of the foregoing changes:

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.^{*}$$

$$\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}.$$

$$\sqrt{8\,a^3b^2} = \sqrt{4\,a^2b^2 \cdot 2\,a} = \sqrt{4\,a^2b^2} \cdot \sqrt{2\,a} = 2\,ab\sqrt{2\,a}.$$

In each of the above the number under the radical sign contains a factor that is an exact power corresponding to the degree of the radical.

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4} \cdot 2} = \sqrt{\tfrac{1}{4}} \cdot \sqrt{2} = \tfrac{1}{2}\sqrt{2}.$$

$$3\sqrt{\tfrac{3}{4}} = 3\sqrt{\tfrac{1}{4} \cdot 3} = 3\sqrt{\tfrac{1}{4}} \cdot \sqrt{3} = 3 \cdot \tfrac{1}{2}\sqrt{3} = \tfrac{3}{2}\sqrt{3}.$$

$$\sqrt[3]{\tfrac{3}{4}} = \sqrt[3]{\tfrac{6}{8}} = \sqrt[3]{\tfrac{1}{8} \cdot 6} = \sqrt[3]{\tfrac{1}{8}} \cdot \sqrt[3]{6} = \tfrac{1}{2}\sqrt[3]{6}.$$

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{\frac{1}{b^2} \cdot ab} = \sqrt{\frac{1}{b^2}} \cdot \sqrt{ab} = \frac{1}{b} \cdot \sqrt{ab}.$$

The above show changes necessary to transform a radical fraction so that the number under the radical sign shall be an integer.

$$\sqrt[4]{9} = \sqrt{\sqrt{9}} = \sqrt{3}.$$

$$\sqrt[6]{9} = \sqrt[3]{\sqrt{9}} = \sqrt[3]{3}.$$

$$\sqrt[10]{32} = \sqrt{\sqrt[5]{32}} = \sqrt{2}.$$

$$\sqrt[4]{4\,a^6b^2} = \sqrt{\sqrt{4\,a^6b^2}} = \sqrt{2\,a^3b}.$$

The foregoing illustrate the changes to be made when the number under the radical sign is an exact power corresponding to some factor of the number denoting the degree of the radical. Thus, in $\sqrt[6]{9}$,

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* The positive even root of a positive real number is called the *Principal root*. The radical sign will be used to indicate the Principal root unless otherwise stated.

> 9 is the second power of 3; the sixth root is the cube root of the square root. The second power of three therefore corresponds to one of the factors of 6, the index of the root.

EXERCISE

113. Simplify the following surds by removing a factor:

1. $\sqrt[3]{16\,x^4}$.
2. $\sqrt{8\,m^3}$.
3. $\sqrt{64\,y^3}$.
4. $\sqrt{28}$.
5. $\sqrt{48}$.
6. $\sqrt[3]{56\,v}$.
7. $\sqrt[3]{-125\,t^4}$.
8. $\sqrt{363\,a^3b^2}$.
9. $(-32\,x^3)^{\frac{1}{2}}$.
10. $\sqrt{5(500)^{\frac{1}{3}}}$.
11. $\sqrt{90\,s^5}$.
12. $\sqrt[3]{192\,x^4y^5z^6}$.
13. $\sqrt{(a+b)(a^2-b^2)}$.
14. $(3\,y^2-18\,y+27)^{\frac{1}{2}}$.
15. $(4\,a^2b-24\,ab^2+36\,b^3)^{\frac{1}{2}}$.
16. $\sqrt{5\,e^2-10\,ed+5\,d^2}$.
17. $\dfrac{a+b}{a-b}\sqrt{(2\,a^2-4\,ab+2\,b^2)^3}$.
18. $\sqrt{(x^2-y^2)(x-y)(x^3+y^3)}$.
19. $\sqrt{R^2-\dfrac{R^2}{2}}$.
20. $\sqrt{25\sqrt{5}+100}$.

Reversing the foregoing process it will be evident that the coefficient of a surd may be written under the radical sign by raising it to the power denoted by the index of the radical and then multiplying the result by the quantity under the radical sign.

Example. $2\sqrt{5}=\sqrt{4}\cdot\sqrt{5}=\sqrt{4\cdot 5}=\sqrt{20}$.

Introduce the coefficient of the following expressions under the radical sign:

21. $5\sqrt{3}$. **22.** $3\sqrt{2}$.

23. $\frac{1}{2}\sqrt{5}$. **24.** $\frac{2}{3}\sqrt{\frac{1}{2}}$.

25. $a\sqrt{b}$. **26.** $2c\sqrt{3e}$.

27. $ab^3\sqrt{abc}$. **28.** $.5e\sqrt{3.4de}$.

29. $\frac{2a}{b}\sqrt{\frac{b}{8a}}$. **30.** $(a-b)\sqrt{\frac{a+b}{a^2-b^2}}$.

Simplify following by extracting a root:

31. $\sqrt[4]{25}$. **32.** $b\sqrt[6]{8a^3}$.

33. $\sqrt[4]{64}$. **34.** $\sqrt[6]{125}$.

35. $\sqrt[6]{81}$. **36.** $\sqrt[4]{144}$.

37. $\sqrt[8]{121x^4y^2}$. **38.** $\sqrt[15]{343a^5b^{10}}$.

39. $\sqrt[12]{216a^6t^3}$. **40.** $\sqrt[6]{4m^3n^2}$.

41. $\sqrt[9]{81e^6k^3}$. **42.** $\sqrt[6]{625r^6n^3}$.

Simplify the following expressions. (Numbers under radical signs must be integers.)

43. $\sqrt{\frac{2}{3}}$. **44.** $\sqrt[4]{\frac{4}{9}}$.

45. $\sqrt[3]{\frac{2}{3}}$. **46.** $\sqrt{4xy}\sqrt{x^2-y^2}\sqrt{\frac{1}{x+y}}$.

47. $\frac{(a+b)^2}{a-b}\sqrt{\frac{a+b}{a-b}}$. **48.** $(a-b)^2\sqrt{\frac{(a+b)^3}{a-b}}$.

49. $\sqrt[5]{\frac{5}{16}}$. **50.** $\frac{\sqrt{a-b}}{\sqrt{a+b}}$.

114. Similar Surds are those having the same root index and the same expressions under the radical signs. *Dissimilar* surds are those that are not similar.

115. Addition and Subtraction of Surds. Similar surds may be added or subtracted by the addition or subtraction of their coefficients. It will often be necessary to simplify surds before adding or subtracting.

Example. $\sqrt{27} + \sqrt{12} + \sqrt{75}.$

$\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$, simplifying,

$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$, simplifying,

$\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$, simplifying,

and $3\sqrt{3} + 2\sqrt{3} + 5\sqrt{3} = 10\sqrt{3}$, adding the coefficients.

EXERCISE

Perform the operations indicated in the following:

1. $\sqrt{12} + 5\sqrt{27} + 3\sqrt{75}.$ 2. $\sqrt[3]{16} + 3\sqrt[3]{54} - \sqrt[3]{128}.$

3. $\sqrt[3]{108} - \sqrt[3]{32} + \sqrt[3]{500}.$ 4. $\sqrt{125} + \sqrt{180} - \sqrt{80}.$

5. $\sqrt{5} + \sqrt{245} - \sqrt{320}.$

6. $7\sqrt{27} - 2\sqrt{75} - 24\sqrt{\frac{1}{12}} - 27\sqrt{\frac{1}{27}}.$

7. $\sqrt{44} - \sqrt{275} + \sqrt{891}.$ 8. $\sqrt[3]{\frac{1}{4}} + \sqrt[3]{\frac{1}{32}} + \sqrt[3]{2}.$

9. $\sqrt{\frac{4}{5}} + \sqrt{\frac{15}{12}} - \sqrt{\frac{1}{3}}.$ 10. $\sqrt[3]{128x} - \sqrt[3]{250x} + \sqrt[3]{54x}.$

11. $(8a^3b + 8a^4)^{\frac{1}{3}} - (8a^4b^3 + 8a^3b^4)^{\frac{1}{3}}.$

12. $\sqrt{x^3yz} - a\sqrt{xyz} + xz\sqrt{\dfrac{b}{xz}}.$

13. $\sqrt{3ab^2} + b\sqrt{75a} + \sqrt{3a(a-9b)^2}.$

14. $\sqrt{3x^3 + 12x^2 + 12x} + \sqrt{27x^3 - 72x^2y + 48xy^2}.$

15. $21\sqrt{\frac{2}{3}} - 5\sqrt{\frac{4}{5}} + 6\sqrt{\frac{27}{6}} - \sqrt{\frac{16}{5}} + \frac{40}{3}\sqrt{\frac{45}{4}}.$

16. $3\sqrt{a^2x + 2abx + b^2x} + 2\sqrt{(a-b)^2x}.$

17. $8\sqrt{a(x+2y)^2} - 3\sqrt{ax^2 + 4axy + 4ay^2}.$

18. $5\sqrt{\frac{3}{5}} - 12\sqrt{\frac{5}{3}} + 6\sqrt{60} - 30\sqrt{\frac{1}{15}}.$

19. $5a\sqrt{12ab^2} - 3b\sqrt{27a^3} + 2\sqrt{300a^3b^2} - 40ab\sqrt{\frac{3a}{4}}.$

20. $\sqrt[6]{(a+b)^3} - n\sqrt[4]{(a+b)^2}.$

116. Equivalent Surds. In the addition and subtraction of surds the necessary condition is that the surds to be added or subtracted be similar. In the multiplication and division of surds it is necessary and sufficient that the surds have the same root index. Transformation of surds of different degrees into surds of the same degree is a necessary step preliminary to the multiplication and division of surds.

Example. Transform the $\sqrt{3}$, $\sqrt[4]{5}$, and $\sqrt[3]{4}$ into equivalent surds having the same root index.

Solution.

$$\sqrt{3} = (3)^{\frac{1}{2}} = (3)^{\frac{6}{12}} = \sqrt[12]{3^6} = \sqrt[12]{729.}$$
$$\sqrt[4]{5} = (5)^{\frac{1}{4}} = (5)^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125.}$$
$$\sqrt[3]{4} = (4)^{\frac{1}{3}} = (4)^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256.}$$

117. Multiplication and Division of Surds.

(1) *Reduce the given expressions, if not of the same degree, to equivalent expressions having the same index.*

(2) *Perform the operations indicated and write the result under the common radical sign or index.*

Example 1. Multiply $\sqrt{3}$ by $\sqrt[3]{4}$.

Solution.

$$\sqrt{3} = (3)^{\frac{1}{2}} = (3)^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$$
$$\sqrt[3]{4} = (4)^{\frac{1}{3}} = 4^{\frac{2}{6}} = \sqrt[6]{16.}$$
$$\sqrt[6]{27} \times \sqrt[6]{16} = \sqrt[6]{432} = \sqrt[6]{216 \cdot 2} = \sqrt[6]{216} \cdot \sqrt[6]{2} = \sqrt{6} \cdot \sqrt[6]{2.}$$

Example 2. Divide $20\sqrt{12}$ by $8\sqrt{3}$.

Solution. $20\sqrt{12} = 20\sqrt{4 \cdot 3} = 20 \times 2\sqrt{3} = 40\sqrt{3}$.

$40\sqrt{3} \div 8\sqrt{3} = 5$.

Example 3. Compare in magnitude $\sqrt{2}$ and $\sqrt[3]{3}$.

Solution. $\sqrt{2} = (2)^{\frac{1}{2}} = (2)^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}$.

$\sqrt[3]{3} = (3)^{\frac{1}{3}} = (3)^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}$.

Since $\sqrt[6]{9} > \sqrt[6]{8}$, it follows that $\sqrt[3]{3} > \sqrt{2}$.*

EXERCISE

Perform the operations indicated and simplify the results.

1. $\sqrt{2} \times \sqrt{4}$.
2. $\sqrt{6} \div \sqrt{9}$.
3. $(4 - \sqrt{6})^2$.
4. $2 \div \sqrt[3]{2}$.
5. $\sqrt[3]{10} \times \sqrt{5}$.
6. $3\sqrt{5} \cdot 4\sqrt{3}$.
7. $\sqrt{\frac{1}{2}} \times \sqrt[3]{\frac{1}{2}}$.
8. $\sqrt{\frac{3}{4}} \div \sqrt{\frac{1}{2}}$.
9. $\sqrt{5} \times \sqrt{3}$.
10. $(6 + 3\sqrt[3]{4})^2$.
11. $6\sqrt{54} \div 3\sqrt{2}$.
12. $\left(\sqrt{\frac{1}{2}} - \sqrt{\frac{2}{3}}\right)^2$.
13. $\sqrt{a^2 - b^2} \div \sqrt{a + b}$.
14. $(5 + \sqrt{7})(5 - \sqrt{7})$.
15. $(a^2 - b^2) \div \sqrt{a + b}$.
16. $(6s - 3\sqrt{t})(s + 2\sqrt{t})^2$
17. $\sqrt[3]{xy} \div \sqrt[4]{y}$.
18. $(5\sqrt{5} + 2)^2$.
19. $\left[2\sqrt[4]{3 - \sqrt[4]{2}}\right]^4$
20. $\left(3\sqrt{x} + \sqrt{2}\right)^2$.
21. $\left(\sqrt{a + \sqrt{b}}\right)\left(\sqrt{a - \sqrt{b}}\right)$.
22. $2\sqrt{3} \div 3\sqrt{2}$.
23. $\left(\sqrt{\frac{3}{4}} + \sqrt{\frac{2}{3}}\right) \div \sqrt{\frac{1}{2}}$.
24. $(x - \sqrt{y})(x^2 + xy + y^2)$.

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* The sign > means "is greater than."

25. $(6\sqrt{18}) \div 3\sqrt{2}$. **26.** $a\sqrt{c} \div a\sqrt{bc}$.

27. $\sqrt{\dfrac{b}{a}} \div \sqrt{\dfrac{c}{a}}$.

28. $(\sqrt{x} - \sqrt[4]{y})(\sqrt{2x} - 2\sqrt[4]{y})$.

29. $(\sqrt[3]{4 + 4\sqrt{3}})(\sqrt[3]{4 - 4\sqrt{3}})$.

30. $-(\sqrt{2} + \sqrt[3]{3})^2$.

118. Rationalization is the process of freeing an expression from radicals. In division by radicals, expecially in dividing by a binomial surd, it often occurs that the quotient has to be expressed in the form of a fraction whose denominator contains the surd divisor. The calculation of the numerical values of such fractions becomes tedious, unless first rationalized.

If the product of a given radical number by another radical is rational, the second is called the *rationalizing factor* of the first.

Thus, $\sqrt{3}$ is the rationalizing factor for $\sqrt{3}$; $\sqrt{5}$ for $\sqrt{5}$; $\sqrt[3]{4}$ for $\sqrt[3]{2}$, since their product is $\sqrt[3]{8} = 2$, which is rational.

Also, $\sqrt[3]{4a^2}$ is the rationalizing factor for $\sqrt[3]{2a}$.

The rationalizing factor for a binomial surd of the form $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$; the latter is called the complement of the given surd.

119. Any Monomial Denominator may be rationalized by multiplying both numerator and denominator of the fraction by the radical contained in the denominator a sufficient number of times.

Example 1. Rationalize the denominator of $\dfrac{2}{\sqrt{3}}$.

Solution. $\dfrac{2}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{\sqrt{9}} = \dfrac{2}{3}\sqrt{3}$.

Example 2. Rationalize the denominator of $\frac{2a}{\sqrt[3]{b}}$.

Solution. $\frac{2a}{\sqrt[3]{b}} \times \frac{\sqrt[3]{b^2}}{\sqrt[3]{b^2}} = \frac{2a\sqrt[3]{b^2}}{\sqrt[3]{b^3}} = \frac{2a}{b}\sqrt[3]{b^2}$.

120. Any Binomial Denominator may be rationalized by multiplying both numerator and denominator of the fraction by the complement surd of the given binomial denominator.

Example. Rationalize the denominator of $\frac{1}{\sqrt{2}-\sqrt{3}}$.

Solution. $\frac{1}{\sqrt{2}-\sqrt{3}} \times \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{2}+\sqrt{3}}{2-3} =$

$$\frac{\sqrt{2}+\sqrt{3}}{-1} = -\sqrt{2}-\sqrt{3}$$

EXERCISE

Rationalize the denominators of the following expressions and reduce to simplest forms:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{3}{\sqrt{5}}$.
3. $\frac{\sqrt{5}}{\sqrt{3}}$.
4. $\frac{4}{\sqrt{3}}$.
5. $\frac{a}{\sqrt{b}}$.
6. $\frac{7t}{5^{\frac{1}{2}}}$.
7. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}-\sqrt{3}}$.
8. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.
9. $\frac{\sqrt{c-d}}{\sqrt{c}-\sqrt{d}}$.
10. $\frac{k-\sqrt{k-n}}{k+\sqrt{k-n}}$.
11. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
12. $\frac{\sqrt{de}}{1-\sqrt{1-d^2e^2}}$.
13. $\frac{3a-4\sqrt{5b}}{3a+4\sqrt{5b}}$.
14. $\frac{\sqrt{3}-3\sqrt{2}}{\sqrt{3}+3\sqrt{2}}$.
15. $\frac{5\sqrt{a}+5\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

121. Imaginaries. As noted in § 108, such expressions as $\sqrt{-5}$, $3\sqrt{-2}$, and $a\sqrt{-b}$, are called imaginaries. Imaginaries and irrational numbers constitute important topics in some of the higher courses in mathematics. Some of the transformations and operations upon imaginaries will be indicated briefly at this time, but their formal study belongs to a more advanced course. These are

(*a*) *Transformations.*

Example.

$$\sqrt{-8} = \sqrt{(8)(-1)} = \sqrt{8}\cdot\sqrt{-1} = 2\sqrt{2}\sqrt{-1} = 2\sqrt{2}\cdot i,$$

where $i = \sqrt{-1}$.

In general, $\sqrt{-b} = \sqrt{b(-1)} = \sqrt{b}\sqrt{-1} = \sqrt{b}\cdot i.$

(*b*) *Addition and Subtraction.*

Example. $\sqrt{-4} + \sqrt{-9} - 2\sqrt{-1}$. Simplify.

$$\sqrt{-4} = \sqrt{4(-1)} = 2i.$$

$$\sqrt{-9} = \sqrt{9(-1)} = 3i.$$

$$-2\sqrt{-1} = -2i,$$

and $2i + 3i - 2i = 3i.$

Hence $\sqrt{-4} + \sqrt{-9} - 2\sqrt{-1} = 3\sqrt{-1}$ or $3i$.

In general,

$$a\sqrt{-b} + c\sqrt{-d} + \sqrt{-f} = a\sqrt{b}\cdot i + c\sqrt{d}\cdot i + \sqrt{f}\cdot i$$
$$= (a\sqrt{b} + c\sqrt{d} + \sqrt{f})i.$$

(*c*) *Multiplication and Division.*

Example. $\sqrt{-1} \times \sqrt{-1} = -1$, since $\sqrt{a} \cdot \sqrt{a} = a$.

That is $(\sqrt{-1})^2 = -1 = i^2$.

Again $\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} = i^3 = i^2 \times i$

$$= -1\sqrt{-1} = -\sqrt{-1}.$$

Also $\sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} \times \sqrt{-1} = i^4 = i^2 \times i^2$

$$= (-1)(-1) = +1.$$

Hence $\sqrt{-4} \times \sqrt{-3} = {}^{*}2\sqrt{-1} \times \sqrt{3}\sqrt{-1}$

$$= 2i \times \sqrt{3}i = 2\sqrt{3} \cdot i^2 = -2\sqrt{3},$$

and $3\sqrt{-5} \times 5\sqrt{-4} = 3\sqrt{5}\sqrt{-1} \times 5\sqrt{4}\sqrt{-1}$

$$= 3\sqrt{5}\sqrt{-1} \times 10\sqrt{-1} = 30\sqrt{5}(-1) = -30\sqrt{5}.$$

Again $\sqrt{-1} \div \sqrt{-1} = 1$, since $\sqrt{a} \div \sqrt{a} = 1$,

and $\sqrt{-20} \div \sqrt{-4} = \sqrt{20}\sqrt{-1} \div \sqrt{4}\sqrt{-1}$

$$= 2\sqrt{5} \cdot i \div 2 \cdot i = \sqrt{5}.$$

In general $\sqrt{-b} \times \sqrt{-c} = \sqrt{b}\sqrt{-1} \times \sqrt{c}\sqrt{-1}$

$$= \sqrt{bc}(-1) = -\sqrt{bc}.$$

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* In the multiplication of imaginaries it is important to observe that $\sqrt{-4} \times \sqrt{-3} \neq \sqrt{(-4)(-3)}$,

for $\sqrt{(-4)(-3)} = \sqrt{12} = 2\sqrt{3}$.

But $\sqrt{-4} \times \sqrt{-3} = 2\sqrt{-1} \times \sqrt{3}\sqrt{-1} = 2\sqrt{3}(-1)$
$= -2\sqrt{3}$,

and $\sqrt{-b} \div \sqrt{-c} = \sqrt{b}\sqrt{-1} \div \sqrt{c}\sqrt{-1}$

$$= \sqrt{b} \div \sqrt{c}, \quad (\text{since } \sqrt{-1} \div \sqrt{-1} = 1) = \frac{\sqrt{b}}{\sqrt{c}}.$$

The foregoing examples are given to show that the fundamental operations are applicable to imaginaries as well as to real numbers.

FOR REVIEW

122. Perform the operations indicated in each of the following:

1. $\sqrt{a^2b} \times \sqrt{a^2b^2c^2}$.
2. $4\sqrt{x^2y^2} \div \sqrt{x^2y^2}$.
3. $5\sqrt{ab} - 10\sqrt{ab} + 12\sqrt{ab}$.
4. $14\sqrt{3} - 6\sqrt{3} + \frac{5}{2}\sqrt{3} + 7\sqrt{3}$.
5. $\sqrt{\frac{a^2}{16}} - \sqrt{\frac{a^2}{25}}$.
6. $\frac{a+b}{a-b}\sqrt{3c} - \frac{a-b}{a+b}\sqrt{3c}$.
7. $(\sqrt{5})(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$.
8. $7\sqrt{27} - \sqrt{75} - 24\sqrt{\frac{1}{12}} - \sqrt{\frac{1}{27}}$.
9. $\sqrt{12x^2 + 60xy + 75y^2} - \sqrt{48x^2 - 72xy + 27y^2}$.
10. $\sqrt{108} \div \sqrt{192}$.
11. $\sqrt{\frac{1}{2}} \div \sqrt[3]{\frac{1}{8}}$.

12. $\sqrt{\dfrac{ab^3}{c^2}} \div (\frac{1}{2}c)\sqrt{a^3b^2 - 4a^2b^2 + 4ab^3}$.

13. $\dfrac{\sqrt{14}}{\sqrt{3}} \div \dfrac{\sqrt{35}}{\sqrt{27}}$.

14. $(a^2 - ab\sqrt{2} + b^2)(a^2 + ab\sqrt{2} + b^2)$.

15. $(3 + \sqrt{2})^2 - (3 - \sqrt{2})^2$.

16. $\sqrt{a^2b} \div \sqrt{3a^3b}$.

17. $\sqrt{\dfrac{3c^2}{4}} \div \sqrt{\dfrac{c^2}{4}}$.

What term must be added to the following to make each a perfect square?

18. $4a^2 + 8ab$.

19. $\dfrac{a^2 - 4ac}{9}$.

20. $9x^2 + 36xy$.

21. Simplify: $\sqrt{(a - b)(a + b)^{-2}} \div \sqrt{a^2 - b^2}$.

Express each of the following in simplest form and with positive exponents only:

22. $\dfrac{c^{-1}d^{-2}c}{b^2t^{-3}}$.

23. $\dfrac{d^2c^3h^4}{a^3b^2t^{-1}}$.

24. b^2ce^{-2}.

25. $\dfrac{a^{-2}b^{-2}c^2}{a^{-4} \cdot b^4}$.

Write with positive exponents and then simplify:

26. $\dfrac{2}{3^{-2}}$.

27. $4^{-\frac{1}{2}}$. (Suggestion: $4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$, ans.)

28. $(.064)^{-\frac{1}{3}}$.

29. $8^{-\frac{2}{3}}$.

30. $(\frac{1}{2})^{-3}$.

31. $(-8)^{-\frac{2}{3}}$.

32. 3^0.

33. $(a \pm b)^0$, if $a \neq b$.

34. $\frac{3^{-1}}{3^{-2} - 3^{-3}}$. (Suggestion: $\frac{3^{-1}}{3^{-2} - 3^{-3}} = \frac{\frac{1}{3}}{(\frac{1}{3})^2 - (\frac{1}{3})^3} = \frac{\frac{1}{3}}{\frac{1}{9} - \frac{1}{27}}$, etc.)

35. $\frac{a^{-3} + b^{-3}}{a^{-4} - b^{-4}}$.

36. $\frac{a^{-2}}{b^{-1} + c^{-1}}$.

37. $\frac{x^{-3} \cdot y^{-3}}{x^{-3} + y^{-3}}$.

38. $\frac{2^{-1} + 3^{-1}}{2^{-5} + 3^{-5}}$.

39. $\frac{4^{-2} \cdot r^{-6} \cdot s^{-4}}{s^{-2} \cdot r^{-4} \cdot t^2}$.

40. $\frac{3^{-}\ (ab)^0}{27^{-2}\, b^{-1}}$.

41. $(\frac{1}{4})^{\frac{1}{2}} \cdot (\frac{1}{2})^{-2} \cdot (\frac{1}{3})^0$.

42. $\frac{\sqrt[3]{a^{-3}} \cdot \sqrt[4]{a^{-4}}}{a^{-5}}$.

43. $(25)^{.5}$.

44. $2^{-1} \div 2^0$.

45. $\frac{1 + x^{-3}}{1 + x^{-1}}$.

Calculate to 2 decimals, the value of the letter in the first member of each of the following equations:

46. $h = \frac{5}{2}\sqrt{3}$.

47. $s = \sqrt{\frac{1}{2}}$.

48. $s = 2\pi r(r + l)$, where $\pi = 3.1416$, $r = \sqrt{10}$, and $l = \frac{1}{2}\sqrt{21}$

49. Solve for t, $s = \frac{1}{2}gt^2$, where $s = 400$ and $g = 32.16$.

50. The area of any triangle is given by the formula, $A = \frac{bh}{2}$ where A denotes area, b any side of the triangle and h the perpendicular distance or altitude from the opposite vertex to b. Express the area A of an equilateral triangle in terms of h.

Express the value of each of the following to three decimals, after first simplifying the expressions:

51. $\sqrt{\frac{2}{3}}$.

52. $\frac{1}{\sqrt{\frac{1}{2}}}$.

53. $\sqrt{\frac{4}{5}}$.

54. $3\sqrt{\frac{1}{27}}$.

CHAPTER VIII

QUADRATIC EQUATIONS. COMPLETING SQUARE AND SOLUTION BY FORMULA—RADICAL EQUATIONS

(a) Quadratic Equations

123. The General Quadratic, as previously noted, is of the form

$$ax^2 + bx + c = 0,$$

in which a, b, and c are known numbers and x represents the unknown number. It should be observed here that:

(a) $a \neq 0$. For, if $a = 0$, the equation reduces to the form $bx + c = 0$ and $x = -\frac{c}{b}\cdot$

(b) If $b = 0$ but a and $c \neq 0$, the equation becomes $ax^2 + c = 0$. Hence, $x^2 = -\frac{c}{a}$ and $x = \pm\sqrt{-\frac{c}{a}}\cdot$ There are, therefore, two roots which we denote x_1 and x_2.* $x_1 = +\sqrt{-\frac{c}{a}}$ and $x_2 = -\sqrt{-\frac{c}{a}}$.

(c) If $c = 0$ but a and $b \neq 0$, the equation becomes $ax^2 + bx = 0$. By factoring, $x(ax + b) = 0$. Here there are two roots, $x_1 = 0$ and $x_2 = -\frac{b}{a}\cdot$

---

* The letters x_2 and x_1 will be used to designate the roots of an equation that has two roots.

(*d*) But in the most general case of the equation $ax^2 + bx + c = 0$, neither a, b, or c equals zero. Since this expression cannot always be factored, other methods of solution must be found. Two methods of solving this general quadratic will be given in the paragraphs which follow.

124. Completing the Square.

Example 1. $x^2 + 4x = 32$, solve for x.

Solution. $x^2 + 4x + 4 = 32 + 4$, adding 4 to both sides.

$x^2 + 4x + 4 = 36$, combining terms of 2d member.

$x + 2 = \pm 6$, taking square root of both sides.

$x = -2 \pm 6$, subtracting 2 from both sides.

Therefore, $x_1 = -2 + 6 = 4$ and $x_2 = -2 - 6 = -8$ are the required roots.

Check.

$(4)^2 + 4(4) = 32$, if $x = 4$. $(-8)^2 + 4(-8) = 32$, if $x = -8$.

$16 + 16 = 32$ $64 - 32 = 32$

$32 = 32$ $32 = 32$

Example 2. $2x^2 - 8x - 24 = 0$, solve for x.

Solution. $2x^2 - 8x = 24$, adding 24 to both sides.

$x^2 - 4x = 12$, dividing both sides by 2.

$x^2 - 4x + 4 = 12 + 4$, adding 4 to both sides.

$x^2 - 4x + 4 = 16$, combining terms of the 2d member.

$x - 2 = \pm 4$, taking square roots of both sides.

$x = 2 \pm 4$, adding 2 to both sides.

Therefore, $x_1 = 2 + 4 = 6$ and $x_2 = 2 - 4 = -2$ are the required roots.

Check.

$2(6)^2 - 8(6) - 24 = 0$, if $x = 6$. $2(-2)^2 - 8(-2) - 24 = 0$, if $x = -2$

$$72 - 48 - 24 = 0 \qquad 8 + 16 - 24 = 0$$
$$0 = 0 \qquad 0 = 0$$

Example 3. $3x^2 + 12x - 18 = 0$, solve for x.

Solution. $3x^2 + 12x = 18$, adding 18 to both sides.

$x^2 + 4x = 6$, dividing both sides by 3.

$x^2 + 4x + 4 = 6 + 4$, adding 4 to both sides.

$x^2 + 4x + 4 = 10$, combining terms of 2d member.

$x + 2 = \pm\sqrt{10}$, taking square roots of both sides.

$x = -2 \pm \sqrt{10}$, subtracting 2 from both sides.

Therefore, $x_1 = -2 + \sqrt{10}$ and $x_2 = -2 - \sqrt{10}$ are the required roots.

Check. $3(-2 + \sqrt{10})^2 + 12(-2 + \sqrt{10}) - 18 = 0$, if $x = -2 + \sqrt{10}$.

$$3(4 - 4\sqrt{10} + 10) - 24 + 12\sqrt{10} - 18 = 0,$$
$$12 - 12\sqrt{10} + 30 - 24 + 12\sqrt{10} - 18 = 0,$$
$$18 - 18 = 0,$$
$$0 = 0.$$

$3(-2 - \sqrt{10})^2 + 12(-2 - \sqrt{10}) - 18 = 0$, if $x = -2 - \sqrt{10}$,

$$3(4 + 4\sqrt{10} + 10) - 24 - 12\sqrt{10} - 18 = 0,$$
$$12 + 12\sqrt{10} + 30 - 24 - 12\sqrt{10} - 18 = 0,$$
$$0 = 0.$$

From the foregoing solutions it will be seen that the process of solving a quadratic equation by completing the square may be summarized as follows:

(1) *Place all terms containing x^2 and x in the first member, and all other terms in the second member. This may be accomplished by the addition and subtraction of terms.*

(2) *Divide both sides of the equation by the coefficient of x^2, if that coefficient is not unity.*

(3) *Add to both sides of the equation the square of half the coefficient of the x term, thereby making the first member a perfect square.*

(4) *Combine and simplify the terms of the second member.*

(5) *Take square roots of both sides of the equation thus simplified, putting the double sign before the term or terms on the right, or second member.*

(6) *Put everything except x of the first member into the second. This may be done by the addition and subtraction of terms.*

(7) *Combine and simplify the terms of the second member, giving due attention to the double sign which* always *precedes one of its terms. This simplified result is the solution desired.*

(8) *Check each solution by substituting the values found for x in the given equation.*

EXERCISE

125. Before solving the equations of this exercise, study them in the light of the observations of § 123.

1. $x^2 - 6x = 91.$

2. $x^2 - 12x = -35.$

3. $y^2 - y - 3 = 0.$

4. $2y^2 + 5y = 0.$

5. $t^2 - 2t - 3\frac{1}{2} = 0.$

6. $x^2 - \dfrac{7x}{12} = 1.$

7. $8v - 24 = 7v^2 + 25.$

8. $4x^2 - 7x = -3.$

9. $5e^2 - 23e + 12 = 0.$

10. $4x^2 + 2x - 1 = 0.$

11. $15y + 4y^2 = 4$.

12. $42 + 2x^2 = -19x$.

13. $k^2 + 10k = -13$.

14. $\frac{1}{3} + \frac{x}{9} = \frac{2}{x}$.

15. $(y - 4)^2 - 3(y - 9) = 15$.

16. $\frac{25t}{2} - \frac{40}{t} - 40 = 0$.

17. $x^2 - \frac{2x}{3} = \frac{1}{2}$.

18. $3.8x = .8 - x^2$.

19. $(2y - 3)^2 = 8y$.

20. $s = vt + \frac{1}{2}gt^2$, solve for t.

PROBLEMS

126. 1. A window in the form of a rectangle surmounted by a semicircle is found to admit the most light when the width and height are equal. If the area of such a window is 64.2744 sq. ft., what is its width? ($\pi = 3.1416$.) (Fig. 34.)

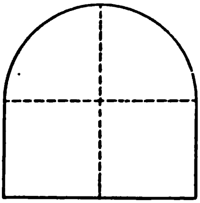


FIG. 34

2. Two automobiles start from Chicago at the same hour, one travelling due east at the rate of 25 miles an hour and the other due north at the rate of 30 miles an hour. How far apart are they at the end of 3 hours?

3. A and B set out at the same time from Chicago and Detroit respectively and travel towards each other. The distance from Chicago to Detroit is 280 miles. A's rate is 10 miles an hour; and B's rate in miles per hour is 4 less than the number of hours required for A and B to meet. Find B's rate.

4. The distance from Chicago to Los Angeles over the Santa Fe route is 2100 miles. The schedule of the California Limited is 5 miles an hour faster than the schedule for the fast mail, and it makes the trip in 12 hours less time. What is the time of the Limited?

5. In a field meet the winner of the 220 yd. hurdles won his race by $\frac{1}{2}$ second, and his speed was $\frac{1}{2}$ yd. greater per second than that of his nearest competitor. What was the winner's time?

6. It was proposed to close a certain street running through a park, thereby saving for park purposes a half acre of ground. The length of the street thus closed was found to be 12 times its width. What was the width of the street?

7. A picture is 12″ by 18″ and is to be framed so that the entire area of both picture and frame shall be twice the area of the picture itself. Find the width of the frame. (Fig. 35.)

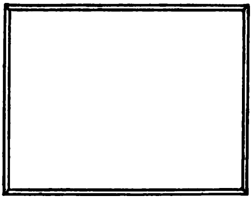


FIG. 35

8. The perimeter of a rectangular piece of ground is 180 feet and its area is 1800 sq. ft. Find the dimensions of the piece of ground.

9. The perimeter of one square field is twice that of another square field and the area of the larger is 4 times that of the smaller. If the number of square feet in the area of the smaller is $1\frac{1}{4}$ times the number of feet in the perimeter of the greater, what is the length of the side of the smaller field?

10. A farmer has a wheat field 120 rods long and 80 rods wide and decides to double its area by adding strips of the same width to all sides of the field. What must be the width of the strip added?

11. A boy has a piece of board 12 inches square. How wide a strip must he cut from each of two adjacent sides to leave a square piece whose area is one-half that of the original piece?

12. A walk extends diagonally across a flower garden 40 feet square. What must be the width of a strip added to each of two adjacent sides to increase the length of the walk 5 feet?

13. The total number of different connections that may be made in a telephone exchange between a 'phone and any other 'phone of the system is $\frac{n(n-1)}{2}$, where n is the number of 'phones belonging to the system. In a certain city the total number of connections is 160,461. How many 'phones in the exchange?

14. A cubic foot of water weighs $62\frac{1}{2}$ lbs. When retained by vertical walls the pressure on them is one-half the product of the depth of the water, the area of the wall below the surface, and the weight of a cubic foot of water. Let p denote pressure, d the depth of the water, and l the length of the wall. Express p in terms of d and l.

15. In the locks of the Panama Canal at Gatun the greatest pressure against the gates is 125,000 lbs. The part of the gate under water being rectangular in form with length $2\frac{1}{2}$ times the depth, find the dimensions of the gate.

16. A cotton dealer bought $10,000 worth of cotton on November 1st. A month later, when the price of cotton had decreased 2 cents a pound, he bought 25,000 lbs. more for $10,000 than on November 1st. What was the price per pound on November 1st?

17. A rectangular piece of tin twice as long as it is wide is made into an open top box by cutting a 3-inch square from each corner and turning up the sides. If the capacity of the box thus formed is 168 cubic inches, find the dimensions of the original piece of tin.

18. If two objects A and B start at the same time from a point P (see Fig. 36) and move in paths intersecting at right angles, the distance between A and B at any time is the hypotenuse of a right triangle whose legs are the respective distances passed over by A and B.

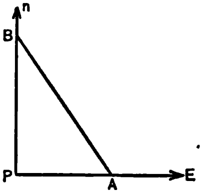


FIG. 36

Suppose that A walks east from a certain city at the rate of 3 miles an hour and B, starting from the same point at the same time, rides north at the rate of 4 miles an hour. In how many hours will they be 25 miles apart?

19. In the preceding problem suppose that A had started one hour later than B, other conditions remaining as

before. In how many hours would they have been 25 miles apart?

20. The length of the pages of a certain book is two inches greater than the width and there is a margin one inch wide. If the printed portion covers $\frac{1}{2}$ the area of the page, find the dimensions of the page.

21. The area of a rectangle is 80 sq. in. and its perimeter is 36 inches. What are its dimensions?

22. A certain number increased by its reciprocal gives $\frac{65}{8}$. Find the number.

23. The square of a certain number added to twice the number itself will give 15. Find the number.

24. The sum of the squares of four consecutive numbers is 54. What are the numbers?

25. A and B together can bind 1000 books in 6 days. When each works separately it is found that A can bind 1000 books in 5 days less time than B. How many books can each bind in a day?

26. A square garden which has an area of 144 sq. ft. has two walks of uniform width running across it at right angles to each other and to the sides of the garden. If the area of the walks is $\frac{1}{2}$ that of the garden itself, find the width of the walks. (Fig. 37.)

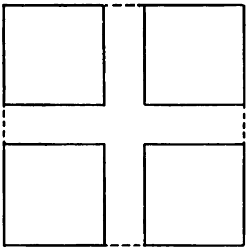


FIG. 37

27. If a bullet is shot upwards with an initial velocity of v, the height s, to which it will rise, is given by the equation, $v = \sqrt{2gs}$. What must be the velocity at the start of a bullet fired upwards if it is to rise to the top of the Metropolitan Life Building, 700 feet? (Assume $g = 32$.)

28. 360 bricks of a certain size are required to cover a piece of paving. But, if the bricks were each 1 inch longer and also 1 inch wider, only 270 of them would be required. What are the dimensions of the bricks?

[Suggestion: The length of a brick is twice its width, unless otherwise specified; assume bricks laid flatwise.]

29. The altitude h of an isosceles triangle divides the triangle into two equal right triangles, (see Fig. 38). If $h = 8$, and $a = c = 10$, find the base and the area of triangle ABC. $\left(\text{Area} = \frac{hb}{2}\cdot\right)$

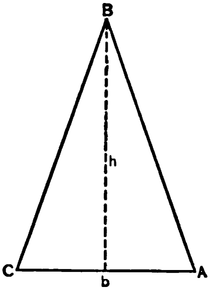


Fig. 38

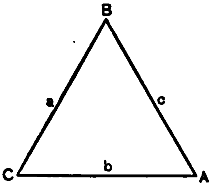


Fig. 39

30. ABC (see Fig. 39) is an equilateral triangle whose area $\left(\frac{hb}{2}\right)$ is $64\sqrt{3}$, and whose perimeter is 48. Find h.

127. Solution by Formula.

Example. $ax^2 + bx + c = 0$, solve for x.

Solution. $ax^2 + bx = -c$, subtracting c from both sides.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}, \text{ dividing both sides by } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2, \text{ adding } \left(\frac{b}{2a}\right)^2 \text{ to both sides.}$$

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$, combining terms in 2d member.

$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}$, taking square roots of both sides.

$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$, subtracting $\frac{b}{2a}$ from both sides.

Therefore,

XXIV $$\mathbf{x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}$$

and $$\mathbf{x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

are the required roots.

The values of x_1 and x_2 as given in the Formula XXIV above hold for all values of a, b, and c. Hence the foregoing expressions may be regarded as formulas for the solution of quadratic equations.

Example. $2x^2 + 6x + 4 = 0$, solve by formula XXIV.

Solution. $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-6 + \sqrt{36 - 32}}{4}$, since $b = 6$, $a = 2$, and $c = 4$,

$= \frac{-6 + 2}{4}$

$= -\frac{4}{4}$

$= -1;$

and
$$\begin{aligned} x_2 &= \frac{-b-\sqrt{b^2-4ac}}{2a} \\ &= \frac{-6-\sqrt{36-32}}{4} \\ &= \frac{-6-2}{4} \\ &= -\frac{8}{4} \\ &= -2. \end{aligned}$$

Therefore, -1 and -2 are the required roots.

Check.
$$\begin{aligned} 2(-1)^2+6(-1)+4 &= 0, \text{ if } x=-1, \\ 2-6+4 &= 0, \\ -6+6 &= 0, \\ 0 &= 0. \\ 2(-2)^2+6(-2)+4 &= 0, \text{ if } x=-2, \\ 8-12+4 &= 0, \\ -12+12 &= 0, \\ 0 &= 0. \end{aligned}$$

EXERCISE

Solve examples in § 125, 1–10, inclusive, by formula.

128. The Discriminant of the Quadratic. The expression b^2-4ac, which occurs as a radical in formula XXIV, may be used to determine the character of the roots; b^2-4ac is called the quadratic discriminant.

It will be observed that:

(1) If b^2-4ac is *positive* the roots are *real* and *unequal.* For, if b^2-4ac is positive, its square root is real and on being added to and subtracted from $-\frac{b}{2a}$,

the result will be two real and unequal numbers. Thus for $2x^2 + 6x + 4 = 0$, $a = 2$, $b = 6$ and $c = 4$, and $b^2 - 4ac = 36 - 32 = 4$. Hence its two roots are real and unequal. Actual solution of this equation showed that its roots were -1 and -2, which thus verifies the formula.

(2) If $b^2 - 4ac = 0$, the roots are *real* and *equal.* For, in this case the radical becomes zero and both roots are equal to the same number, $-\frac{b}{2a}$.

Thus for the equation, $x^2 - 6x + 9 = 0$, $a = 1$, $b = -6$, and $c = 9$. Hence $b^2 - 4ac = 36 - 36 = 0$. The actual solution of this equation shows that $x = 3$ is the only root, or rather than both roots equal the same number 3 and so are equal, thus verifying the conclusion given above.

(3) If $b^2 - 4ac$ is *negative*, the roots are *complex.*[*] For, in this case we have the square root of a negative which is imaginary. And on being combined with $-\frac{b}{2a}$ the result will generally be in the form $a + bi$, which is complex.

Thus for the equation $x^2 - 6x + 16 = 0$, $a = 1$, $b = -6$, and $c = 16$. Hence, $b^2 - 4ac = 36 - 64 = -28$; $x_1 = \frac{6}{2} + \frac{1}{2}\sqrt{-28} = 3 + \frac{1}{2}\sqrt{-28}$; and $x_2 = \frac{6}{2} - \frac{1}{2}\sqrt{-28} = 3 - \frac{1}{2}\sqrt{-28}$. These results verify the conclusion reached with the discriminant.

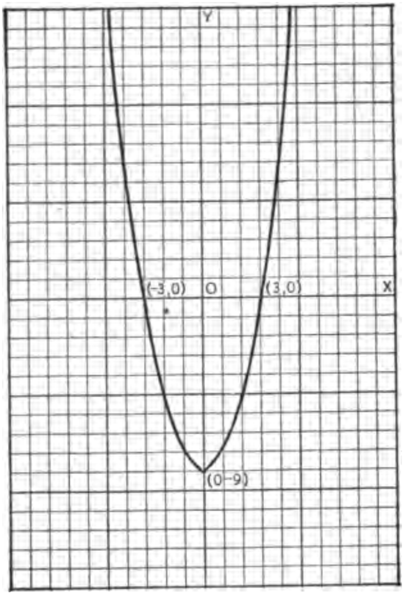
---

[*] A complex number consists of a real and an imaginary part. Thus $2 + \sqrt{-3}$ and $1 - \sqrt{-5}$ are complex numbers. The general form of such numbers is $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. In case a is zero the complex number reduces to a pure imaginary. Thus $2 + \sqrt{-3} = 2 + \sqrt{3} \cdot \sqrt{-1} = a + bi$, when $a = 2$, $b = \sqrt{3}$, $i = \sqrt{-1}$. If $a = 0$, $i\sqrt{3}$ or $\sqrt{-3}$ is imaginary.

EXERCISE

Determine from the discriminant the character of the roots of the following:

1. $x^2 - 7x + 4 = 0$.
2. $y^2 - 6y + 9 = 0$.
3. $y^2 + 6y + 9 = 0$.
4. $4x^2 - 16x + 16 = 0$.
5. $4x^2 - 3x + 1 = 0$.
6. $8x^2 - 4x = 13$.
7. $6y^2 - 3y = 1$.
8. $x^2 + 8x + 7 = 0$.
9. $3x^2 + x = 0$.
10. $x^2 + 2x + 4 = 0$.



x	y
0,	-9.
1,	-8.
2,	-5.
3,	0.
4,	7.
5,	16.
-1,	-8.
-2,	-5.
-3,	0.
-4,	7.

FIG. 40

129. The Graph of a Quadratic in x.

Example 1. Construct the graph of $x^2 - 9 = 0$.

Solution. Put $y = x^2 - 9$. Then if $x = 0$, $y = -9$, etc. See table.

These points are plotted and connected by a smooth curve in Fig. 40. From this curve it appears that the given equation has two distinct solutions, $x = 3$, and $x = -3$, since the curve crosses the x-axis at the points (3, 0) and (– 3, 0).

Example 2. Construct the graph of $x^2 - 6x + 9 = 0$ and show from the graph that this equation has equal roots.

Solution. Put $y = x^2 - 6x + 9$. Then if $x = 0$, $y = 9$, etc. See table.

These points are plotted and connected by the smooth curve in Fig. 41. A study of this curve shows that it does not cross the x-axis at all, merely touching the axis at the point (3, 0). In such cases the two points of intersection of the graph with the axis have merged into a single point, that is, the roots are not distinct but equal.

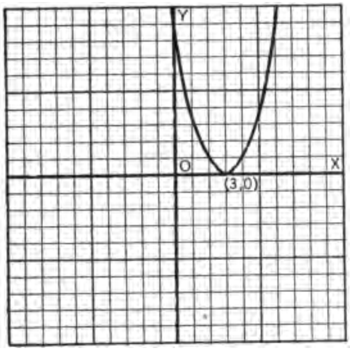


FIG. 41

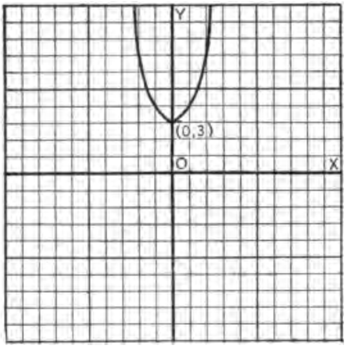
x	y
0,	9.
1,	4.
2,	1.
3,	0.
4,	1.
5,	4.

Example 3. Construct the graph of $x^2 + 3 = 0$, and show from the graph that this equation has no real solution, that is, its roots are imaginary.

Solution. Put $y = x^2 + 3$. Then if $x = 0$, $y = 3$, etc. See table.

These points are plotted and connected by the smooth curve in Fig. 42. It is seen also from the foregoing values for x and y

that no value of x, either positive or negative, will give a zero or a negative value for y; that is, all the points on this graph lie above the x-axis. Hence on this graph there can be no intersection of the graph with the x-axis, and hence no real solution; that is, the roots are imaginary.



x	y
0,	3.
1,	4.
2,	7.
− 1,	4.
− 2,	7.

FIG. 42

From the foregoing examples it will be seen that to construct the graph of a quadratic containing a single unknown represented by x,

(1) *Put the given expression equal to y. Since the resulting equation contains two letters, x and y, it will clearly be satisfied for an indefinite number of pairs of values. x and y are variables.*

(2) *Assume different values for x and compute the corresponding values of y.*

(3) *Plot the points corresponding to these pairs of values for x and y.* (Use cross-section paper.)

(4) *Join these points by a smooth curve; this curve will be the graph of the given quadratic.*

It will also be seen that the abscissas of the points of intersection of the graph with the x-axis are the roots of the quadratic equation in x. Hence:

(1) An equation whose graph intersects the x-axis in two points has two real and distinct roots.

(2) An equation whose graph meets the x-axis in one point has equal roots.

(3) An equation whose graph does not intersect the x-axis at all cannot have any real roots; that is, its roots are complex or imaginary.

EXERCISE

Solve each of the following equations graphically:

1. $x^2 + 8x + 7 = 0.$
2. $4x^2 - 4x = 20.$
3. $3x^2 + x = 0.$
4. $9y^2 + 6y + 1 = 0.$
5. $25y^2 - 4 = 0.$
6. $x^2 + 2x + 4 = 0.$
7. $x^2 - 7x + 4 = 0.$
8. $3x^2 + 4x - 5 = 0.$
9. $4x^2 - 3x = -1.$
10. $8x^2 - 4x = 13.$

130. Equations in Quadratic Form. An equation consisting of three or more terms, two of which contain the unknown number, and have the exponent of the unknown number in one of these terms twice what it is in the other, is said to be quadratic in form.

Thus, $x^6 + 4x^3 + 7 = 0$; and $y^{\frac{5}{2}} - 3y^{\frac{5}{4}} + 8 = 0$, are quadratic in form.

The solution of such equations is usually obtained by putting the lowest power of the unknown number in the given equation equal to the first power of some other letter; the second power of this letter will then represent the highest power of the unknown number in the given equation.

Example. $y^6 - 4y^3 - 5 = 0$, solve for y.

Solution. Let $x = y^3$.

Then $x^2 - 4x - 5 = 0$, substituting x for y^3 in the given equation.

$(x - 5)(x + 1) = 0$, factoring the quadratic in x.

$x - 5 = 0$, or $x + 1 = 0$, equating factors to zero.

Hence, $x = 5$, or $x = -1$,

and, since $y^3 = x$, as assumed above,

$y^3 = 5$, and therefore, $y = \sqrt[3]{5}$.

Again, since $y^3 = x = -1$, $y = \sqrt[3]{-1} = -1$.

Therefore, $y = \sqrt[3]{5}$ and $y = -1$ are two * of the solutions of the given equation.

EXERCISE

Using the method illustrated in the foregoing example, find two roots of the following equations:

1. $y^4 - 17y^2 + 16 = 0$.
2. $4z^6 - 13z^3 + 9 = 0$.
3. $x^5 - 5x^{\frac{5}{2}} - 2 = 0$.
4. $27x^3 + 19x^{\frac{3}{2}} - 8 = 0$.
5. $3y^{\frac{2}{3}} - 4y^{\frac{1}{3}} = 4$.
6. $9y^{\frac{1}{2}} - 13y^{\frac{1}{4}} + 4 = 0$.

---

* It should be remarked in passing that there are other roots also to this equation; to find such roots, however, is beyond the scope of this course.

7. $x^{\frac{4}{3}} - x^{\frac{2}{3}} - 1 = 0.$ 8. $(x - 5)^4 - 3(x - 5)^2 = 4.$

9. $2(2x - 3)^{\frac{1}{2}} + 5(2x - 3)^{\frac{1}{4}} - 7 = 0.$

10. $2\left(x + \frac{1}{x}\right)^2 - 9\left(x + \frac{1}{x}\right) + 10 = 0.$

11. $4(s - 5)^2 - 12(s - 5) = -9.$

12. $(x + 2)^5 - (x + 2)^{\frac{5}{2}} = 2.$

13. $(3y^7 - 5y^{\frac{7}{2}} - 2 = 0.$ 14. $9t^{\frac{3}{2}} - 15t^{\frac{3}{4}} + 6 = 0.$

15. $3y^8 - 11y^4 + 10 = 0.$ 16. $4v^4 + 5 = 21v^2.$

17. $\left(x + \frac{2}{x}\right)^2 + 4\left(x + \frac{2}{x}\right) = 21.$

18. $\sqrt{(7x - 6)^2} - 4\sqrt[3]{7x - 6} + 4 = 0.$

19. $(x + 2) + 3\sqrt{x + 2} - 4 = 0.$

20. $3(2x^2 - 1)^2 + (2x^2 - 1) - 2 = 0.$

(b) Radical Equations

131. A Radical Equation is an equation containing one or more radical expressions.

Thus, $\sqrt{2x + 5} = 2$; $\sqrt{x + 1} - 2 = \sqrt{x - 1}$, and $(x + 1)^{\frac{2}{3}} = 16$ are radical equations. It should be noticed that only the positive sign precedes the radical.

Example 1. $x^{\frac{3}{2}} = 8.$ Solve for x.

Solution. $\sqrt{x^3} = 8$, definition of fractional exponent.

$x^3 = 64$, raising both sides to the second power.

$x = 4$, taking cube roots of both sides.

Check. $(4)^{\frac{3}{2}} = 8$, substituting the value of x in the given equation.

$$(\sqrt{4})^3 = 8,$$
$$(2)^3 = 8,$$
$$8 = 8.$$

Therefore, 4 is a root of the equation $x^{\frac{3}{2}} = 8$.

Example 2. $\sqrt[4]{2x + 11} = \sqrt{5}$. Solve for x.

Solution. $2x + 11 = 25$, raising both sides to the 4th power.

$2x = 25 - 11$, subtracting 11 from both sides.

$2x = 14$, collecting terms.

$x = 7$, dividing both sides by 2.

Check. $\sqrt[4]{14 + 11} = \sqrt{5}$,

$$\sqrt{25} = 5,$$
$$5 = 5.$$

Therefore, 7 is a root of the equation $\sqrt{2x + 11} = 5$.

Example 3. $\sqrt{2x + 7} = \sqrt{x} + 2$. Solve for x.

Solution. $2x + 7 = x + 4\sqrt{x} + 4$, squaring both sides.

$x + 3 = 4\sqrt{x}$, simplifying.

$x^2 + 6x + 9 = 16x$, squaring both sides.

$x^2 - 10x + 9 = 0$, simplifying.

$(x - 9)(x - 1) = 0$, factoring.

Therefore, $x_1 = 9$, and $x_2 = 1$, equating factors to zero.

Check. $\sqrt{18 + 7} = \sqrt{9} + 2$, substituting 9 for x.

$$\sqrt{25} = 3 + 2.$$
$$5 = 5.$$

$\sqrt{2 + 7} = 1 + 2$, substituting 1 for x.

$$\sqrt{9} = 1 + 2.$$
$$3 = 3.$$

Therefore 9 and 1 are roots of equation $\sqrt{2x + 7} = \sqrt{x} + 2$.

Example 4. $\sqrt{x-2}+\sqrt{4x+1}=3$. Solve for x.

Solution. (1) $\sqrt{4x+1}=3-\sqrt{x-2}$, subtracting $\sqrt{x-2}$ from both sides of the given equation.

(2) $4x+1=9-6\sqrt{x-2}+(x-2)$, squaring (1).

(3) $3x-6=-6\sqrt{x-2}$, simplifying (2).

(4) $x-2=-2\sqrt{x-2}$, dividing both sides of (3) by 3

(5) $x^2-4x+4=4x-8$, squaring both sides of (4).

(6) $x^2-8x+12=0$, simplifying (5).

(7) $(x-6)(x-2)=0$, factoring (6).

Therefore, $x=6$ and $x=2$, equating factors to zero.

Check. $\sqrt{6-2}+\sqrt{24+1}=3$, substituting 6 for x.

$\sqrt{4}+\sqrt{25}=3$,

$2+5=3$, which of course is not true.

Therefore, 6 is not a root of the equation, that is, 6 is an extraneous* root.

$\sqrt{2-2}+\sqrt{8+1}=3$, substituting 2 for x.

$0 \quad \sqrt{9}=3$,

$3=3$.

Hence, 2 is a root of the equation.

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* An extraneous root is a value of the unknown number that has been introduced at some point in the process of solving the equation. On going back over the solution of example 4 to find at what point the extraneous root 6 was introduced, it will be seen that this root value 6 satisfies equations (5) and (6) which are therefore equivalent, but fails to satisfy equation (4) and the equations preceding. Hence equations (1), (2), (3), and (4) are not equivalent to (5) and (6); and it was in deriving equation (5) from (4) that this value 6 was introduced. Equation (5) was derived from (4) by squaring (4), so that squaring the members of an equation is one of the ways in which extraneous roots *may be* introduced. In examples 1, 2, and 3 above

132. Summary. As illustrated in the examples of § 131, the solution of an equation containing radicals is obtained as follows:

(1) *It is to be understood that when no sign precedes the radical* ONLY POSITIVE *values of such expressions are to be considered.*

(2) *When an equation contains but one radical term, this radical should constitute one member of the equation.*

(3) *When an equation contains but two radicals, it is to be transformed so that one radical shall appear in each member before raising to the required power.*

(4) *When necessary to repeat the process of raising to a power, simplify each side of the equation as much as possible before doing this.*

(5) *Solve the equation freed from radicals according to methods already given.*

(6) *Check each result and reject any result that does not satisfy the given equation.*

EXERCISE

133. In solving the following equations point out such as have extraneous roots:

1. $(2y+1)^{\frac{1}{2}}=5.$
2. $\sqrt{s-11}+\sqrt{s}-11=0.$
3. $\sqrt{3x+24}-\sqrt{3x}=1.$

no such roots were found, however, so that squaring does not always lead to the introduction of extraneous roots. As frequently stated before, the only sure way of determining whether a number is a root or not is to see if it satisfies the equation.

4. $\sqrt{v - 2a} - \sqrt{v - 6a} = \sqrt{2v - 5a}$.

5. $2\sqrt{x + 3} - 3\sqrt{10 - x} = 0$.

6. $1 + \sqrt{x - 1} = 3$.

7. $\sqrt{x + 3} = \sqrt{2x - 10}$.

8. $\sqrt[3]{x - 1} = 2$.

9. $(3 - x)^{\frac{1}{4}} - (3 + x)^{\frac{1}{2}} = 0$.

10. $y - \sqrt{4y^2 - 25y + 6} = 6$.

11. $\sqrt{x - a} - \sqrt{x - b} = b$.

12. $\sqrt{y - 1} + \sqrt{2y - 3} = 7$.

13. $\sqrt{x - 2} + \sqrt{4x + 1} = 3$.

14. $\sqrt{3x^2 - 2} = 8 - x$.

15. $\sqrt{x + 3} + \sqrt{x - 4} - \sqrt{4x - 3} = 0$.

CHAPTER IX

FRACTIONS—RATIO AND PROPORTION

(a) Fractions

134. A Fraction in Algebra is defined as an indicated operation in division. The dividend is written above a horizontal line and is called the *numerator* of the fraction. The divisor is written below the same horizontal line and is called the *denominator* of the fraction. Numerator and denominator taken together are called the *terms* of the fraction.

Thus the expression $\frac{a}{b}$, in which a and b represent either polynomials or numbers (excepting zero), is an algebraic fraction. a is the numerator, b the denominator; a and b together are the terms of the fraction.

The notation and rules of fractions as treated in arithmetic are used in algebra with but slight modifications.

The *laws of exponents* and the *laws of signs* for integral numbers hold also for fractions.

The fundamental theorem of fractions, however, may be be expressed as follows:

XXV
$$\mathbf{\frac{a}{b} = \frac{ma}{mb}},$$

where a, b, and m are any numbers whatever, zero excepted.

In words, this means that *the numerator and the denominator of a fraction may be multiplied by the same number or divided by the same number without changing the value of the fraction.*

135. Common Factors. The numerator and denominator of a fraction often have one or more factors in common. In the simplest cases these common factors can be determined by inspection. In other cases the common factors can be determined only after the factors of numerator and denominator have been written out. The product of all the factors common to two or more polynomials is the *highest common factor* of these polynomials.

Thus, 3, 2, a and b are the common factors of $12ab$ and $18ab$; and $6ab$, the product of 3, 2, a and b is, therefore, the highest common factor (designated H. C. F.), of $12ab$ and $18ab$. Likewise, $x - y$ is the H. C. F. of $x^2 - 2xy + y^2$ and $x^2 - y^2$.

EXERCISE

Find the H. C. F. of the following:

1. 36, 42 and 48.

2. 108, 216 and 378.

3. $12abc$, $16ab$ and $24bc$.

4. $21xy$, $35yz$ and $56xyz$.

5. $b - c$ and $ab - ac$.

6. $6(m - n)^2$, $9(m - n)^3$, $12(m - n)^4$.

7. $(x - y)^3$, $x^2 - y^2$, $x^2 - 2xy + y^2$.

8. $8a^3(a^4 - b^4)$, $a^2(a^3 + b^3)$, $3a^4(a^2 - b^2)$.

9. $x^3 - y^3$, $x^4 - y^4$, and $x^5 - y^5$.

10. $4p(x^2 - y^2)^2$ and $2pq(x^6 - y^6)$.

136. Reduction to Lowest Terms. Since, Formula XXV,

$$\frac{ma}{mb} = \frac{a}{b},$$

it follows that both numerator and denominator of a fraction may be divided by their highest common factor without changing the value of the fraction.

Example. Reduce $\frac{18\,a^3bc}{12\,ab^2c^2}$ to its lowest terms.

Solution. H. C. F. of $18\,a^3bc$ and $12\,ab^2c^2$ is $6\,abc$.

$\frac{18\,a^3bc}{12\,ab^2c^2} = \frac{6\,abc\,(3\,a^2)}{6\,abc\,(2\,bc)} = \frac{3\,a^2}{2\,bc}$, dividing both terms by $6\,abc$.

EXERCISE

Reduce the following fractions to their lowest terms:

1. $\frac{24\,a^3b^4c^2}{60\,a^2bc}$.

2. $\frac{x^2 - y^2}{(x - y)^2}$.

3. $\frac{a^2 - 2\,ab + b^2}{4\,z(a - b)}$.

4. $\frac{m^2 + 2\,mn + n^2}{m + n}$.

5. $\frac{s^3 - t^3}{3(s - t)}$.

6. $\frac{3(b - a)(a + c)}{3(b - a)(a - c)}$.

7. $\frac{x^2 - 2\,xy + y^2 - z^2}{x - y + z}$.

8. $\frac{6(a - b)^2 - 6\,k}{9(a - b - k)^2}$.

9. $\frac{a(a^2 + a - 12)}{ab(3\,a^2 - 13\,a + 12)}$.

10. $\frac{36\,m^2 + 97\,mn + 36\,n^2}{9\,m^2 + 13\,mn + 4\,n^2}$.

137. Common Multiples. A common multiple of two or more expressions is an expression that is exactly divisible by each of the given expressions. The lowest common multiple of several expressions is the one of lowest degree that is exactly divisible by each of them.

In the addition and subtraction of fractions it is frequently necessary to transform the fractions given into equivalent fractions which have a common denominator. This common denominator is the common multiple of all the denominators of the fractions to be added or subtracted. The lowest common multiple is designated L. C. M.

Example. Find the L. C. M. of $24abc$, $18abc$, and $12abc$.

Solution. Factoring the given expressions, we have

$$24abc = 2 \cdot 2 \cdot 2 \cdot 3 \cdot a \cdot b \cdot c = 2^3 \cdot 3 \cdot a \cdot b \cdot c.$$

$$18abc = 3 \cdot 3 \cdot 2 \cdot a \cdot b \cdot c = 2 \cdot 3^2 \cdot a \cdot b \cdot c.$$

$$12abc = 2 \cdot 2 \cdot 3 \cdot a \cdot b \cdot c = 2^2 \cdot 3 \cdot a \cdot b \cdot c.$$

Since the L. C. M. is to be exactly divisible by each of the given expressions, it must contain 2^3 as a factor. If the L. C. M. contains 2^3 as a factor, it will also contain 2^2 and any lower power of 2 as factors. Similarly, the L. C. M. must contain 3^2 as a factor. Also, a, b, and c will be factors of the L. C. M. Therefore, the L. C. M. of the given expressions is $2^3 \cdot 3^2 \cdot a \cdot b \cdot c = 72abc$.

From the foregoing example it will be seen that the method of finding the L. C. M. of two or more expressions is covered by the following rule: *Resolve each of the expressions into its prime factors. Multiply together each of the different prime factors, taking each factor the greatest number of times it occurs in any of the given expressions.*

EXERCISE

Find the L. C. M. of:

1. $9a$, $15ab$, $18ab$.
2. $15mn$, $54mn^2$, $90m^2n$.
3. $x^3 - a^3$, $x^2 - a^2$, $x - a$.
4. $a^3 - c^3$, $a^2 - c^2$, $c - a$.
5. $a^3 - b^3$, $a^2 - b^2$, $a - b$.
6. $1 - y^2$, $y - 1$, $y - 2$, $-y^2$.
7. $x + a$, $3x^2 + 3ax + 3a$.
8. $2x^2 + x$, $4x - 1$, $4x + 1$.
9. $x^3 + y^3$, $x + y$, $xy - x - y$.
10. $3x^2 - 5x + 2$, $4x^3 - 4x^2 - x + 1$.

In each of the following examples transform the fractions into others having a lowest common denominator:

11. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}.$
12. $\frac{m+n}{3mn^2}, \frac{m-n}{2m^2n}.$
13. $\frac{3y}{x+3}, \frac{5}{x^2+5x+6}.$
14. $\frac{a}{a-b}, \frac{b}{b-a}, \frac{c}{-a-b}.$
15. $\frac{3a}{c^2-d^2}, \frac{4}{c+d}.$
16. $\frac{2y+5}{y^2-1}, \frac{2y-5}{y^3-3y^2+2y}.$
17. $\frac{x-y}{a^2-b^2}, \frac{x+y}{a+b}.$
18. $\frac{xy}{a-b}, \frac{xy(a^2-ab+b^2)}{a^3-b^3}.$
19. $\frac{a}{(a-b)(b-c)}, \frac{b}{(b-a)(c-a)}.$
20. $\frac{x^2-y^2}{(x-y)(x+y)}, \frac{(x-y)(x+y)}{x^3-y^3}.$

138. The Signs of a Fraction. There are three algebraic signs to be considered in connection with every fraction; the sign of the numerator, the sign of the denominator, and the sign of the fraction as a whole.

Thus, the fraction $-\dfrac{-(x-y)}{3(x+y)}$ has minus for the sign of its numerator, plus for the sign of its denominator, and minus for the sign of the whole fraction.

The following rules relating to the signs of a fraction will be found of much value:

(1) *The signs of all the terms of a fraction may be changed without changing the value of the fraction, since this is equivalent to multiplying both the numerator and the denominator by the same number,* -1. (Formula XXV, $m = -1$.)

Thus,
$$\frac{-x+y}{-a+b} = \frac{x-y}{a-b}.$$

(2) *The signs of all the terms of the numerator, or of all the terms of the denominator, may be changed by changing the sign of the fraction, since this is equivalent to multiplying one of the terms of a fraction by* -1.

Thus,
$$-\frac{-(m+n)}{x-y} = \frac{m+n}{x-y}.$$

(3) *The signs of an even number of factors of the numerator, or of the denominator of a fraction, or of the numerator and the denominator together, may be changed without affecting the value of the fraction, since multiplying by* -1 *an even number of times is the same as multiplying by* 1. Thus,

$$\frac{(a-b)(m-n)(d-c)}{(a-b)(n-m)(c-d)} = \frac{(a-b)(n-m)(c-d)}{(a-b)(n-m)(c-d)} = 1.$$

For, changing the signs of two numerator factors does not change the sign of the product.

(4) *Changing the signs of an odd number of factors of the numerator or of the denominator or of both those terms, changes*

the sign of the fraction; since multiplying by -1 *an odd number of times means the same thing as multiplying by* -1 *once.* Thus,

$$\frac{(a-b)(c-d)(p-q)}{(b-a)(d-c)(q-p)} = -\frac{(b-a)(d-c)(q-p)}{(b-a)(d-c)(q-p)} = -1.$$

139. Mixed Expressions. An expression that consists of an integral and a fractional part, separated by a plus or a minus sign, is called a mixed expression.

Thus, $M - \frac{m}{n}$ and $X + \frac{y}{z}$, since each consists of an integer and a fraction, are mixed expressions.

It should be observed that such an expression as $M\frac{m}{n}$ is not a mixed expression, since in algebra the omission of a sign between two quantities indicates a product. In arithmetic $5\frac{2}{3}$ means the sum of 5 and $\frac{2}{3}$; but to denote the same thing in algebra, the plus sign must be placed between the integer and the fraction; that is, the expression $M + \frac{m}{n}$, not $M\frac{m}{n}$, signifies the same as $5\frac{2}{3}$ does in arithmetic. It is very important for the student to remember this distinction as to fractions in what follows.

140. Reduction of Mixed Expressions. Mixed expressions in algebra may be written in fractional form.

Example. Reduce $2x + \frac{y}{x-y}$ to fractional form.

First, multiplying $2x$ by $(x-y)$, gives $2x^2 - 2xy$.
Second, adding the numerator, y, to $2x^2 - 2xy$ gives
$2x^2 - 2xy + y$.

Third, writing this sum as the numerator and $x - y$ as the denominator, gives $\frac{2x^2 - 2xy + y}{x - y}$ as the fraction equivalent to the given mixed expression.

From the foregoing example we have the following rule for the reduction of a mixed expression to one fractional in form:

Multiply the integral part of the expression by the denominator of the fraction and add to this product the numerator of the fraction. Write the sum thus obtained over the denominator of the fractional part.

EXERCISE

Reduce the following to fractional form:

1. $2xy + \frac{1}{x - y}$.
2. $p + q - \frac{2}{(p + q)^2}$.
3. $x + y - \frac{2xy}{x + y}$.
4. $a + b + \frac{c}{a^2 - b^2}$.
5. $a - b + \frac{a - b}{a + b}$.
6. $2 - \frac{4xy}{(x + y)^2}$.
7. $1 - \frac{2xy}{(x + y)^2}$.
8. $b - c - \frac{(b + c)^2}{b^2 - c^2}$.
9. $a + \frac{b^2}{(a - b)^2}$.
10. $c + d - \frac{c}{1 + c - d}$.

141. Reduction of Improper Fractions. If the numerator of a fraction is of the same degree or of a higher degree than the denominator, it may be divided by the denominator. The result will be an integer if the division is exact, and a mixed expression if not exact. Fractions that may be thus divided are called *improper fractions.*

Thus, in the fraction of the preceding paragraph, dividing the numerator, $2x^2 - 2xy + y$, by the denominator,

$(x - y)$, gives a quotient of $2x$ and a remainder, y, which with its own proper sign is written over the denominator as a fraction. The process of changing an improper fraction into a mixed expression is thus the reverse of the process of changing a mixed expression into a fraction.

EXERCISE

Express as integers or mixed expressions:

1. $\dfrac{a^2 + ab + b^2}{a}$.
2. $\dfrac{a^2 - 2b^2}{a + b}$.
3. $\dfrac{x^2 - 2xy - y^2}{x - y}$.
4. $\dfrac{x^3 - 2xy^2 + y^3}{x - y}$.
5. $\dfrac{a^3 + b^3}{a + b}$.
6. $\dfrac{p^2 - rp + r^2}{r - p}$.
7. $\dfrac{a^3 + 3a^2b + 3ab^2 + 2b^3}{a + b}$.
8. $\dfrac{27a^3 - 64b^6}{3a - 4b^2}$.
9. $\dfrac{x^3 - y^3}{x + y}$.
10. $\dfrac{x^4 + x^2y^2 + y^4}{x^2 + y^2 + xy}$.

142. Addition and Subtraction of Fractions.

Illustrative Examples.

1. In arithmetic, $\frac{2}{5} + \frac{3}{5} + \frac{4}{5} = \frac{9}{5}$; and in like manner in algebra, $\frac{a}{y} + \frac{b}{y} + \frac{c}{y} = \frac{a + b + c}{y}$.

2. In arithmetic, $\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}$; and in like manner in algebra,

$$\frac{x}{x - y} + \frac{2x}{x^2 - y^2} = \frac{x(x + y)}{x^2 - y^2} + \frac{2x}{x^2 - y^2} = \frac{x^2 + xy + 2x}{x^2 - y^2}.$$

The L. C. M. of the denominators, $x - y$ and $x^2 - y^2$ is $x^2 - y^2$, which is the denominator of the second fraction. Multiplying both terms of the first fraction by $x + y$, the quotient obtained by dividing $(x^2 - y^2)$ by $(x - y)$ gives $\frac{x^2 + xy}{x^2 - y^2}$. Adding the numerators of the two fractions which now have a common denominator, gives $\frac{x^2 + xy + 2x}{x^2 - y^2}$ as the sum of the fractions.

3. Simplify $\frac{5ab}{a^2 - b^2} - \frac{10ac}{2(a - b)}$.

The L. C. M. of $a^2 - b^2$ and $2(a - b)$ is $2(a^2 - b^2)$, which is consequently the lowest commom denominator of the denominators of the given fractions. Multiplying both terms of the first fraction by 2 and of the second fraction by $(a + b)$ gives $\frac{10ab}{2(a^2 - b^2)}$ and $\frac{10a^2c + 10abc}{2(a^2 - b^2)}$, equivalent fractions having a common denominator. Subtracting the numerators of these fractions gives,

$$\frac{10ab - 10a^2c - 10abc}{2(a^2 - b^2)} = \frac{10a(b - ac - bc)}{2(a^2 - b^2)} = \frac{5a(b - ac - bc)}{a^2 - b^2}$$

as the difference between the given fractions.

From the foregoing examples it will be seen that the process of addition and subtraction of fractions may be stated as follows:

(1) *If the fractions have a common denominator, they may be added or subtracted by adding or subtracting their numerators and writing the result over the denominator.*

(2) *If the given fractions do not have a common denominator, they must be reduced to equivalent fractions having a common denominator, after which their numerators may be added or subtracted as before.*

From the foregoing illustrative examples it will be seen that the lowest common denominator of two or more fractions is the L. C. M. of their denominators.

EXERCISE

Simplify:

1. $\frac{3}{2y}+\frac{2}{y}-\frac{1}{3y}$.

2. $\frac{2}{3b}-\frac{3}{4by}+\frac{1}{y}$.

3. $\frac{5}{2mn}-\frac{2}{3mx}-\frac{1}{nx}$.

4. $\frac{a+2c}{2ac}-\frac{6a-1}{6a^2}$.

5. $\frac{2b^2y+3}{4by^2}+1-\frac{3b+y}{6y}$.

6. $\frac{x+1}{2}-\frac{x-1}{2}$.

7. $y-\frac{3y+1}{8}+\frac{1-3y}{6}$.

8. $\frac{m+1}{m}-3+\frac{7-3m}{3m^2}$.

9. $\frac{m-1}{m+1}+\frac{m+1}{m-1}$.

10. $\frac{3x-4y}{2}-\frac{2x-y-z}{3}+\frac{15x-4z}{12}$.

11. $\frac{x+1}{(x-1)^2}+\frac{2x}{x^2-1}$.

12. $\frac{(m+n)^2}{4(m-n)^2}-\frac{2-m}{m-n}$.

13. $\frac{1}{y^2+y}-2+\frac{2y^2}{y^2+y}$.

14. $\frac{1}{3m-3}+\frac{1}{2m+2}+\frac{m-5}{6m-6}$.

15. $\frac{2}{2t-1}+\frac{3}{4t+2}+\frac{7t}{4t^2-1}$.

16. $\frac{a}{a^2-1}+2-\frac{a-1}{a+1}-\frac{a-2}{a-1}$.

17. $\frac{x}{x+y} - \frac{xy}{(x+y)^2} - \frac{xy}{(x+y)^3}$.

18. $\frac{-a+2}{2a^2+a-1} - \frac{a-3}{4a^2-1} + \frac{2a+5}{2a^2+3a+1}$.

19. $1 - \frac{2}{a-1} + a - \frac{3a-1}{a+1} - \frac{2a-5}{2}$.

20. $\frac{2}{a+4} - \frac{a-3}{a^2-4a+16} + \frac{a^2}{a^3+64}$.

143. Multiplication of Fractions. The product of two or more fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the given fractions.

Thus, in arithmetic, $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$;

in algebra, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

144. Division of Fractions. The quotient of one fraction divided by another is a fraction whose numerator is the quotient of the numerators of the given fractions and whose denominator is the quotient of the denominators of the given fractions.

Thus, in arithmetic, $\frac{4}{9} \div \frac{2}{3} = \frac{2}{3}$;

in algebra, $\frac{a^2}{b^2} \div \frac{a}{b} = \frac{a}{b}$.

EXERCISE

145. Perform the operations indicated, expressing results in simplest form.

1. $\frac{8\,am^2}{27\,cn} \cdot \frac{9\,cn^3}{16\,am^3}$.
2. $\frac{2\,x}{b} \cdot \frac{3\,ab}{2\,c} \cdot \frac{3\,ac}{a}$.
3. $\frac{10\,x^2y^3}{2\,ab^2} \div \frac{4\,xy}{8\,abc}$

Suggestion: invert the terms of the divisor and multiply as in arithmetic.

4. $\frac{y^2+y-2}{y^2-7\,y} \cdot \frac{y^2-13\,y+42}{y^2+2\,y}$.

Suggestion: factor each of the expressions.

5. $\frac{m^2-4\,m+3}{m^2-5\,m+4} \cdot \frac{m^2-9\,m+20}{m^2-10\,m+21}$.
6. $\frac{8a}{a^3-b^3} \div \frac{4\,a}{a^2+ab+b^2}$.
7. $\frac{x^3-y^3}{x^2-xy+y^2} \cdot \frac{x^3+y^3}{x^2+xy+y^2} \cdot \frac{x}{x-y}$.
8. $\frac{y^2-y-12}{y^2-9} \cdot \frac{y^2-2y-3}{y^2-2y-8} \div \frac{y^2+y}{y+2}$.

Suggestion: invert terms of last fraction and multiply all three fractions together.

9. $\frac{a^2+2\,a-15}{a^2+8\,a-33} \div \frac{a^2+9\,a+20}{a^2+7\,a-44}$.
10. $\frac{a^3-3\,a^2b+3\,ab^2-b^3}{a^2-b^2} \cdot \frac{3}{2\,ab-2\,b^2}$.
11. $\frac{b^2-bx-b+x}{b^2-3\,b+2} \cdot \frac{b^2-8}{b^3-x^3} \div \frac{8-b^3}{b^2+bx+x^2}$,

12. $\dfrac{a^2-(b-c)^2}{c^2-(b-a)^2} \div \dfrac{(b-c)^2-a^2}{(a-b)^2-c^2}$.

13. $\dfrac{6c^2-c-2}{4c^2+4c-3} \cdot \dfrac{2c^2-5c-12}{9c^2+6c-8} \div \dfrac{2c^2-7c-4}{6c^2+5c-4}$.

14. $\left(\dfrac{4x^2-5x+1}{8x^2-2x-2}\right)\left(\dfrac{3x^2-4x+1}{3x^2-1-6x^2}\right) \div \left(\dfrac{1-3x+2x^2}{5x-1-6x^2}\right)$.

15. $\dfrac{m^2+n^2}{n+m} \div \dfrac{m^4+2m^2n^2+n^4}{m^2n-4mn^2}$.

16. $\left(\dfrac{2a}{x}\right)^4 \div \left(\dfrac{-6ac}{5x}\right)^3\left(\dfrac{3c^3}{10ax}\right)^2$.

17. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right) \div \left(\dfrac{a^6+b^6}{a^3b^3}\right)\left(a^2-b^2+\dfrac{b^4}{a^2}\right)\left(\dfrac{1}{ab^2}\right)$.

18. $\dfrac{-6t^3}{12a^2} \div \dfrac{8a^2t}{15} \cdot \dfrac{-8at^2}{27}$.

19. $\dfrac{b^2-by}{p-q} \div \dfrac{b^4-y^4}{(b-y)^2}$.

20. $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)^2 \div \left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)$.

146. Complex Fractions. A complex fraction is a fraction whose numerator or denominator or both consist of a fraction. A complex fraction may be considered as an indicated division of one fraction by another.

A complex fraction is regarded as being simplified when the indicated division has been carried out.

Example 1. Simplify: $\dfrac{\dfrac{ab}{a-b}}{\dfrac{b}{a-b}}$.

Solution. $$\frac{\frac{ab}{a-b}\times(a-b)}{\frac{b}{a-b}\times(a-b)} = \frac{ab}{b},$$ multiplying both numerator and denominator by $(a - b)$, Formula XXV.

$= a$, dividing both numerator and denominator by b.

Example 2. Simplify: $\dfrac{c+\frac{c}{d}}{1+\frac{1}{d}}$.

Solution. $$\frac{c+\frac{c}{d}}{1+\frac{1}{d}} = \frac{\frac{cd+c}{d}}{\frac{d+1}{d}},$$ expressing numerator and denominator as single fractions.

$$\frac{cd+c}{d} \div \frac{d+1}{d},$$ expressing as division.

$$\frac{c(d+1)}{d} \times \frac{d}{d+1},$$ inverting terms of divisor, also factoring numerator of the first fraction.

$= c$, dividing both terms by common factors.

EXERCISE

Simplify the following:

1. $\dfrac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}$.

2. $\dfrac{\frac{x^2-y^2}{x}}{\frac{x+y}{y}}$.

3. $\dfrac{\dfrac{a^3 - b^3}{a + b}}{\dfrac{a^3 - b^3}{a}}$.

4. $\dfrac{1 + \dfrac{a}{b}}{1 + \dfrac{b}{a}}$.

5. $\dfrac{\dfrac{a}{b} + \dfrac{a + b}{a - b}}{\dfrac{b}{a} + \dfrac{a - b}{a + b}}$.

6. $\dfrac{\dfrac{x - 2}{x - 3} - \dfrac{x + 3}{x + 4}}{\dfrac{1}{x - 1} - \dfrac{7}{x - 3}}$.

7. $\dfrac{\dfrac{x + 1}{x - 1} + \dfrac{x - 1}{x + 1}}{\dfrac{x + 1}{x - 1} - \dfrac{x - 1}{x + 1}}$.

8. $\dfrac{\dfrac{b + c}{c} + \dfrac{c}{b + c}}{\dfrac{1}{b} + \dfrac{1}{c}}$.

9. $\left(x^4 - \dfrac{1}{x^4}\right) \div \left(x - \dfrac{1}{x}\right)$.

10. $\dfrac{a - b - \dfrac{a + b}{a - b}}{b - a + \dfrac{a^2}{a + b}}$.

11. $\dfrac{3}{2n + 1} + \dfrac{3}{2n - 1} - \dfrac{5n^2}{8n^3 + 1} - \dfrac{5n^2}{8n^3 - 1}$.

12. $\dfrac{1 - e^2 + \dfrac{e^4}{1 - e^2}}{1 - e^2}$.

13. $\left(\dfrac{1 + x}{1 + x^2} - \dfrac{1 + x^2}{1 + x^3}\right) \div \left(\dfrac{1 + x^2}{1 + x^3} - \dfrac{1 + x^2}{1 - x^4}\right)$.

14. $\dfrac{1}{x - 1} - \dfrac{1}{x - 2} + \dfrac{1}{x - 3} - \dfrac{1}{x - 4}$.

Suggestion: Combine the first two fractions, then the last two, and then combine the results.

15. $(e^2 + e^{-2})\left(1 + \dfrac{e^2 - 2 + e^{-2}}{4}\right)$.

16. $\left(\frac{a^2}{b^2} + 1 + \frac{b^2}{a^2}\right) \div \left(\frac{a}{b} - 1 + \frac{b}{a}\right).$

17. $(2 - xy + y^2) \div \frac{2y}{x + y}.$

18. $\left(1 + \frac{4\,xy}{(x - y)^2}\right) \div \left(1 + \frac{xy + y^2}{x^2 - y^2}\right).$

19. $\dfrac{\dfrac{a^2 - b^2 - c^2}{2\,bc} - 1}{\dfrac{a^2 + b^2 - c^2}{2\,ab} + 1}.$

20. $\frac{(c + d)^2}{c^3 + d^3} \cdot \frac{c^2 - cd + d^2}{c^2 + 2\,cd + d^2}.$

(b) Ratio and Proportion

147. Ratio. A fraction has been defined as an indicated operation in division. A fraction is also called a *ratio*; when $\frac{a}{b}$ is written in the form $a : b$ it is read "*The ratio of a to b.*" The first number of a ratio is called the *antecedent;* the second, *consequent.*

148. A Proportion is an equality of fractions or ratios. Thus, if $\frac{4}{6} = \frac{2}{3}$, the four terms of these two fractions are called a proportion.

149. The Form of a Proportion. A proportion may be written in the fractional form, as $\frac{a}{b} = \frac{c}{d}$; or, it may be written thus: $a : b = c : d$. The first and fourth terms of a proportion are called the extremes; the second and third terms, the means. Thus, a and d are extremes, b and c means, in the above proportion.

150. Operations Allowable upon Ratios and Proportions. Since a ratio is a fraction, whatever operations may be performed upon a fraction may also be performed upon a ratio. Hence, Formula XXV holds also for a ratio. And, since a proportion is an equality of two or more fractions, whatever operations may be performed upon an equation may also be performed upon a proportion; that is, we may add the same number to both sides; subtract the same number from both sides; multiply or divide both sides by the same number.

151. The Fundamental Property of a Proportion. If

$a : b = c : d$, then

XXVI $$\mathbf{ad = bc.}$$

This means that in any proportion *the product of the means equals the product of the extremes.*

152. Standard Transformations in a Proportion. If

$a : b = c : d$, then $\frac{a}{b} = \frac{c}{d}$; and $\frac{a}{b} + 1 = \frac{c}{d} + 1$.

adding 1 to both sides.

Simplifying,

XXVII $$\mathbf{\frac{a+b}{b} = \frac{c+d}{d}};\quad \text{also}\quad \mathbf{\frac{a+b}{a} = \frac{c+d}{c}}.$$

This means that the four numbers a, b c and d are in proportion by *composition.*

Again, if $$\frac{a}{b} = \frac{c}{d},$$

it follows that $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

subtracting 1 from both sides.

Simplifying,

XXVIII $$\mathbf{\frac{a-b}{b} = \frac{c-d}{d}};\quad \text{also,}\quad \mathbf{\frac{a-b}{a} = \frac{c-d}{c}}.$$

This means that the four numbers a, b, c, and d are in proportion by *division.*

If unity be divided by each side of the equation $\frac{a}{b} = \frac{c}{d}$, neither of these terms being zero, the result will be

$$\frac{1}{\frac{a}{b}} = \frac{1}{\frac{c}{d}},$$

or, simplifying,

XXIX $$\mathbf{\frac{b}{a} = \frac{d}{c}}.$$

This means that the numbers a, b, c, and d are in proportion by *inversion.*

Again, since $ad = bc$, it follows that

$$\frac{ad}{dc} = \frac{bc}{dc},$$ dividing both sides by dc.

Simplifying,

XXX
$$\frac{\mathbf{a}}{\mathbf{c}} = \frac{\mathbf{b}}{\mathbf{d}}.$$

This means that the numbers a, b, c, and d are in proportion by *alternation.*

153. The Solution of a Proportion is based upon the fundamental property, expressed in Formula XXVI, that

$$ad = bc.$$

From this it follows that if three of the four terms of the proportion are known, the equation resulting from putting the product of the extremes equal to the product of the means may be solved for the fourth term in the same manner as any other equation containing a single unknown quantity.

Thus, in $\frac{a}{b} = \frac{c}{x}$, suppose a, b, and c represent known numbers, while x represents the unknown number.

Then $ax = bc$, fundamental property.

And $x = \frac{bc}{a}$, dividing both sides by a.

In the proportion, $a : b = b : c$, solving this for b as shown above for x, $b^2 = ac$; and hence $b = \sqrt{ac}$. b in such case is called a *mean proportional* to a and c.

EXERCISE

Simplify the following ratios by writing them as fractions and then reducing the fractions to their simplest forms:

1. 5 : 20.
2. 15 : 3.
3. 4.5 : 9.
4. $\frac{3}{4} : \frac{1}{2}$.

5. $16ab : 24a$.

6. $(1-x)^2 : (1-x)$.

7. $\left(2+\frac{1}{x}\right) : \left(4-\frac{1}{x^2}\right)$.

8. $\frac{1}{x-3} : \frac{1}{x^2-5x+6}$.

Find the value of the unknown in the following proportions:

9. $4 : 7 = 16 : y$.

10. $5 : x-3 = 7 : 2x+6$.

11. $\frac{1}{y} : 2 = 3 : \frac{9}{2}$.

12. $3 : y = 7 : 9$.

13. $\frac{4}{3} : 2 = y : .5$.

14. $40 : 5 = 8 : x$.

15. $a : 5 = \frac{1}{2} : 7$.

16. $\frac{2}{3} : \frac{1}{x} = 4 : \frac{3}{5}$.

17. $(x-2) : (x-1) = 4 : 5$.

18. $(y-1) : 4 = 5 : (y-2)$.

19. $(m-1)^2 : (m-1) = 8 : 1$.

20. $2(k-5) : 4 = 6 : 4(k-5)$.

Find a mean proportional between the following numbers:

21. 4 and 9.

22. 16 and 12.

23. $\frac{1}{3}$ and $\frac{1}{27}$.

24. $(x-y)^2$ and 4.

25. a^2b^2 and a^4b^4.

26. $\frac{1}{2}$ and $\frac{1}{8}$.

27. $\sqrt{2}$ and $\sqrt{5}$.

28. m and n.

If $\frac{a}{b} = \frac{b}{c}$, c is called a *third proportional* to a and b.

Find a third proportional to the following numbers:

29. 1 and 4.

30. 16 and 4.

31. $\frac{1}{3}$ and $\frac{1}{27}$.

32. $\frac{1}{2}$ and $\frac{1}{8}$.

Find the fourth proportional to the following numbers:

33. 1, 2 and 3.

34. 7, 14 and 5.

35. a, a^2 and a^3.

36. $a + b$, $a - b$ and $a^2 - b^2$.

37. x^2, x^3 and x.

38. $\dfrac{1}{a}$, $\dfrac{1}{a^2}$ and $\dfrac{1}{a^3}$.

Write each of the following expressions as a proportion:

39. $4 \cdot 5 = 2 \cdot 10$.

40. $3 \cdot 6 = 2 \cdot 9$.

41. $mp = nq$.

42. $mn = 9$.

43. $mn = m + n$.

44. $x - y = 3 \cdot 4$.

Take the following proportions by alternation:

45. $\frac{2}{3} = \frac{6}{9}$.

46. $4 : 5 = 10 : y$.

47. $\dfrac{b}{c} = \dfrac{d}{2}$.

48. $\dfrac{x}{y} = 1$.

Take each of the following proportions by inversion:

49. $\dfrac{4}{5} = \dfrac{m}{n}$.

50. $\dfrac{1}{2} = \dfrac{b}{c}$.

51. $\frac{3}{2} = \frac{9}{6}$.

52. $s : t = u : v$.

53. Take each of the last four proportions by composition; then by division; and finally by both composition and division at the same time.

PROBLEMS LEADING TO PROPORTIONS

154. 1. The sides of a triangle are 4, 5 and 6 inches respectively. The side 6 is divided into two parts which have the same ratio as the other two sides of the triangle. Find these parts. (Fig. 43.)

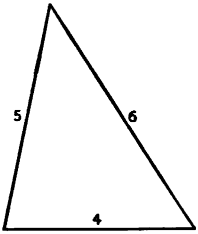

FIG. 43

2. A flagstaff casts a shadow 36 feet long at the same time that a man 6 feet tall casts a shadow 43 inches long. What is the height of the flagstaff?

3. The perimeter of a triangle is 81 inches and its sides are to each other as 2 : 3 : 4. Find the sides of the triangle. (Fig. 44.)

4. Two men start at the same time and travel in the same direction, their rates of travel being to each other as 2 : 3. In 5 hours they are 100 miles apart. Find their rates of travelling.

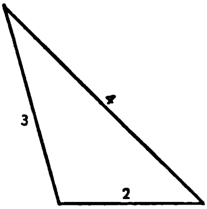

FIG. 44

5. Each of three herdsmen had the same number of cattle and all were fed from a supply of grain furnished by the first two, one of whom furnished $\frac{7}{8}$ as much as the other. The third man paid the others \$30 for his share of the grain. How should this be divided between the other two?

6. A teeter board is 12 feet long. Where must the support be placed in order that an 80-pound boy at one end may balance a 60-pound boy on the other end and a 40-pound boy 5 feet from the other end.

7. A bar 30 inches long is balanced by a 6-lb. weight at one end and a 10-pound weight at the other end. Find the position of the support.

8. The perimeter of a triangle is 63 inches and the longest side is to the shortest as 3 : 2, while the other side is a mean proportional between the other two. Find the sides of the triangle. (Fig. 45.)

In geometry if two figures are similar, the sides of the first taken in any order are proportional to the sides of the second taken in the same order.

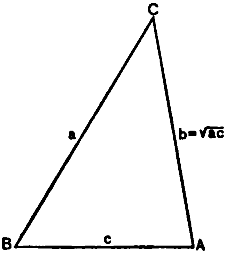

FIG. 45

9. The sides of a triangle are 5, 6 and 7 inches respectively, and in a similar triangle the side corresponding to 5 is 10. Find the other sides. (Fig. 46.)

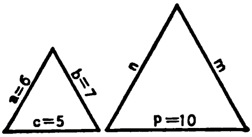

FIG. 46

10. The sides of a triangle are 2, 3 and 4. The perimeter of a second triangle similar to the first is 108. What are the sides of the second triangle? (Fig. 47.)

In geometry, the areas of similar figures have the same ratio as the squares of their corresponding sides.

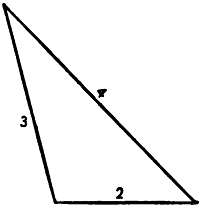

FIG. 47

11. The area of a triangle whose base is 10 inches is 50 sq.ins. If the area of a second triangle similar to the first is 200 sq.ins., what is the base of the second triangle?

12. The area of a square is 64. What is the ratio of the side of this square to the side of another square whose area is 9 times the area of the first? (Fig. 48.)

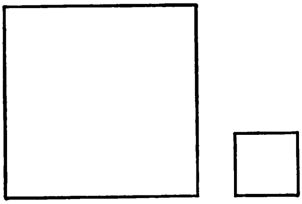

FIG. 48

13. Wishing to know the height of a church steeple down the street, a man noticed that by holding a 12-inch ruler vertically in front of his face at the distance of two feet, the ruler and the steeple both subtended the same angle. Assuming that the lower end of the ruler was in the same horizontal line as his eye and the base of the steeple, and that the distance down the street to the foot of the steeple was 600 feet, how tall was the steeple? (See Fig. 49.) In the figure, b = 12 inches = 1 foot, DC = 2 feet and CA = 600 feet. x = ?

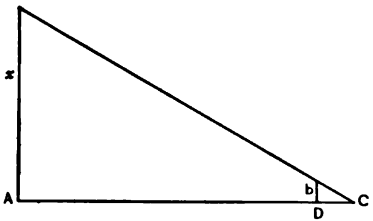

FIG. 49

14. Wishing to know the distance between two points on opposite sides of a mountain, stakes were set at A, C and D, and another at E so that DE would be parallel to AB, and the following measurements taken: c = 6000 feet, e = 4000 feet, and b = 7000 feet. How far is B from A? (See Fig. 50.)

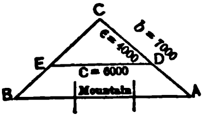

FIG. 50

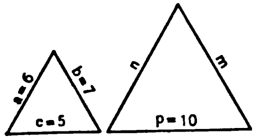

FIG. 51

15. How many proportions can you make with a, b, c, p, m and n, when these quantities are the sides of similar triangles as in Fig. 51?

16. In a circle the perpendicular from any point in the circumference to the diameter is a mean proportional between the two segments of the diameter. In Fig. 52 DC is perpendicular to AB and meets AB in C which is 4 units from A and 16 units from B. How long is DC?

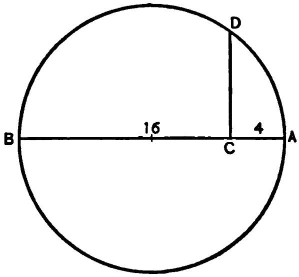

FIG. 52

MISCELLANEOUS EXERCISES FOR REVIEW

155. 1. Change $-1+\dfrac{(a^2+b^2)^2}{4a^2b^2}$ to an improper fraction.

2. Simplify in the shortest possible manner:

$$\frac{1}{x+2}+\frac{1}{x-3}+\frac{1}{x+3}+\frac{1}{x-2}.$$

3. Simplify:

$$\frac{1}{a-\dfrac{1}{a}}+\frac{1}{a+\dfrac{1}{a}}-\frac{2}{a^3-\dfrac{1}{a^3}}+\frac{b}{ab+\dfrac{b}{a}}.$$

Perform the operations indicated in the following:

4. $\left[\left(\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}\right)+\left(\dfrac{m^2+n^2+2n^2}{m-n}\right)\right]\div\left(\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}\right).$

5. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}+x^2+y^2+\dfrac{2x^2y^2+2y^4}{x^2-y^2}\right)$

$$\div\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right).$$

6. $\dfrac{x+y+z}{x+y}+\dfrac{z^2}{(x+y)^2}+\dfrac{(x+y)^3}{(x+y)^3-z^3}+1+\dfrac{z}{x+y}.$

7. Simplify:

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$$

8. $\dfrac{xy}{(y+z)(x+z)}+\dfrac{yz}{(z+x)(x+y)}+\dfrac{xz}{(x+y)(y+z)}$

$$+\frac{2xyz}{(x+y)(y+z)(x+z)}.$$

9. $(z + \frac{1}{3})(z^2 - \frac{1}{9})(z - \frac{1}{3})$.

10. $\dfrac{a+p}{a-p} - \dfrac{a-p}{a+p} + \dfrac{a+p}{a-p} + \dfrac{a-p}{a+p}$.

11. $\left(\dfrac{x^2}{y^2} - 1\right)\left(\dfrac{x}{x-y} - 1\right)\left(\dfrac{x}{y} - 1\right) \div \left(\dfrac{x+xy}{x^2+xy+y^2}\right)$.

12. $\dfrac{x^2}{x^2-y^2} - \dfrac{y^2}{x^2+y^2} + \dfrac{(x^2-y^2)^2}{(x^2-y^2)^2+(x^2+y^2)^2}$.

13. $z^3 + \dfrac{1}{z^3} - 3\left(\dfrac{1}{z^2} - z^2\right) - 4\left(z + \dfrac{1}{z}\right) \cdot \left(z + \dfrac{1}{z}\right)$.

14. $\left(\dfrac{z^3}{a^3} - \dfrac{z}{a} + \dfrac{a}{z} - \dfrac{a^3}{z^3}\right) \div \left(\dfrac{z}{a} - \dfrac{a}{z}\right)$.

15. $\left(\dfrac{m}{m-n} - \dfrac{n}{m+n}\right) \div \left(\dfrac{m^2}{m^2+n^2} + \dfrac{n^2}{m^2-n^2}\right)$.

16. Find the value of $a^3 + b^3 + c^3 - 3abc$, if $a = .5$, $b = .02$ and $c = 2.5$.

17. Using the same values for a, b and c as in Example 16, find the value of $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

If $\dfrac{a}{b} = \dfrac{c}{d}$, prove the following:

18. $a - b : a = c - d : c$.

19. $a - b : b = c - d : d$.

20. $a - b : a + b = c - d : c + d$.

21. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, show that $\dfrac{a+c+e}{b+d+f} = \dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$.

In words this means that in a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.

(Suggestion: Put $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$. Then $a = bk$, $c = dk$ and $e = fk$. Why? Next add these equalities.)

22. If $ad = bc$, show that $a : b = c : d$. In words show that if the product of two numbers a and d, equals the product of two other numbers, b and c, one pair of these numbers may be made the means and the other pair the extremes of a proportion.

23. The capacities of two tanks are to each other as $x : y$, and their combined capacity is $x^3 + y^3$ gallons. Find the capacity of each.

24. The ratio of the annual savings of two boys is 2 : 3. If in 5 years their combined savings amount to $1000, how much did each save in a year?

25. The sides of two squares are to each other as 2 : 3 and their combined area is 1300 sq.ft. Find the sides of the squares.

26. Let the side of a square be 10 inches and let the middle point of one side be joined to the ends of the opposite side. What is the length of each of the two equal sides of the isosceles triangle thus formed?

27. Let s denote the side of a square and d its diagonal. Express (1) s in terms of d; (2) d in terms of s.

28. Let the diagonal of a square be 20 inches and let s represent one of the sides. Express s (1) as a radical in its simplest form; (2) as a decimal to two decimal places.

29. Let the side of a square be 12 inches and let d represent the diagonal. Express d (1) as a radical in its simplest form; (2) as a decimal to two decimal places.

30. Find the altitude and the area of an equilateral triangle whose side is a feet.

31. What are the altitude and sides of an equilateral triangle whose area is $\frac{16}{3}\sqrt{3}$?

32. Find the altitude and the side of an equilateral triangle whose area is $a^2\sqrt{3}$.

Simplify each of the following:

33. $\frac{pq}{p+q}\left(\frac{p}{q}-\frac{q}{p}\right)$.

34. $\left(\frac{k}{k-n}-1\right)\div\left(1-\frac{k}{k+n}\right)$.

35. $\frac{a^5-b^5}{a-b}\cdot\frac{a+b}{(a-b)^2}\cdot\frac{a^4-b^4}{(a+b)^3}\cdot\frac{a^2-b^2}{a^2+b^2}$.

36. $\dfrac{\frac{x^2}{y^2}-\frac{y^2}{x^2}}{x+\frac{y^2}{x}}$.

37. $\dfrac{\frac{a}{1+a}+\frac{1-a}{a}}{\frac{a}{1+a}-\frac{1-a}{a}}$.

38. $\dfrac{\frac{1+a}{a}}{1-\frac{1}{a^2}}$.

39. $\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}$.

40. $(2\sqrt{3}+3\sqrt{5})\div\sqrt{15}$.

CHAPTER X

FRACTIONAL EQUATIONS—LITERAL LINEAR EQUATIONS—LINEAR SYSTEMS IN THREE OR MORE UNKNOWNS

(a) Fractional Equations

156. A Fractional Equation is an equation containing fractional expressions some of whose denominators contain the unknown. Thus, $\frac{5}{x} - 3 = \frac{1}{2}(x - 1)$ and $\frac{x - 2}{x - 1} = \frac{3x - 5}{x + 2} - 8$ are fractional equations.

157. The Solution of Fractional Equations when the denominators are monomials is readily obtained by multiplying both sides by the L. C. M. of the denominators, thus freeing the equation of fractions; and then solving the resulting equation as in Chapter IV. When the denominators are polynomials, the solution depends on the form of the polynomial denominators. The illustrative solutions given below will be sufficient to indicate the method to be pursued in solving the examples that follow.

158. Illustrative Problems.

1. Solve $\frac{1}{3} + \frac{2}{5x} = \frac{13}{15} - \frac{2}{3x}$;

Solution. $5x + 6 = 13x - 10$, multiplying both sides by $15x$, the L. C. M. of the denominators.

$5x - 13x = -10 - 6$, subtracting $13x$ and 6 from both sides.

$-8x = -16$, collecting terms.

Therefore, $x = 2$, dividing both sides by -8.

Check. $\frac{1}{3} + \frac{2}{10} = \frac{13}{15} - \frac{2}{6}, \quad \frac{16}{30} = \frac{16}{30}.$

2. Solve $\frac{5}{1-x} + \frac{6}{1+x} = \frac{7}{1-x^2}.$

Solution. $5(1+x) + 6(1-x) = 7$, multiplying both sides by $(1-x^2)$ the L. C. M. of the denominators.

$5 + 5x + 6 - 6x = 7$, removing the parentheses.

$5x - 6x = 7 - 5 - 6$, subtracting 5 and 6 from both sides.

$-x = -4$, collecting terms.

Therefore, $x = 4$, multiplying both sides by -1.

Check. $\frac{5}{1-4} + \frac{6}{1+4} = \frac{7}{1-16}.$

$$-\frac{5}{3} + \frac{6}{5} = -\frac{7}{15}, \quad -\frac{7}{15} = -\frac{7}{15}.$$

3. Solve $\frac{1}{x-2} - \frac{1}{x-3} = \frac{1}{x-4} - \frac{1}{x-5}.$

Solution. $\frac{x-3-(x-2)}{(x-2)(x-3)} = \frac{x-5-(x-4)}{(x-4)(x-5)}$, combining terms on each side.

$\frac{-1}{x^2-5x+6} = \frac{-1}{x^2-9x+20}$, simplifying terms.

$x^2 - 9x + 20 = x^2 - 5x + 6$, multiplying both sides by the L. C. M. of the denominators.

$x^2 - x^2 - 9x + 5x = 6 - 20$, subtracting $x^2 - 5x$ and 20 from both sides of the equation.

$-4x = -14$, combining terms on both sides.

Therefore, $x = 3\frac{1}{2}$, dividing both sides by -4.

Check.

$$\frac{1}{\frac{3}{2}} - \frac{1}{\frac{1}{2}} = \frac{1}{-\frac{1}{2}} - \frac{1}{-\frac{3}{2}}$$

$$\tfrac{2}{3} - 2 = -2 - (-\tfrac{2}{3}),$$

$$-\frac{4}{3} = -\frac{4}{3}.$$

EXERCISE

159. Solve each of the following equations for the unknown quantities and check solutions:

1. $\dfrac{x+1}{x-3} + \dfrac{x-5}{x} = 2.$

2. $\dfrac{2(x+3)}{x+3} + \dfrac{4}{x+5} = 2.$

3. $\dfrac{h+18}{h+14} = \dfrac{h+3}{h+2}.$

4. $\dfrac{n-2}{n-4} = \dfrac{n+16}{n+2}.$

5. $\dfrac{y^2+2y-3}{3y^2+4y+1} = \dfrac{y-1}{y+1}.$

6. $\dfrac{3e-20}{e-2} = \dfrac{3e-12}{e+3}.$

7. $\dfrac{k-1}{k+3} = \dfrac{k-4}{k-1}.$

8. $\dfrac{1}{x-3} + \dfrac{2}{x+3} = \dfrac{-3}{x^2-9}.$

9. $\dfrac{1}{x-3} = \dfrac{x-6}{x+3} + \dfrac{x}{x+3}.$

10. $\dfrac{5}{7y+5} + \dfrac{1}{8} = 0.$

11. $\dfrac{3}{x-2} = \dfrac{5}{x^2-25} + \dfrac{3x}{x^2-25}.$

12. $\dfrac{y+2}{y+3} + \dfrac{y+3}{y+2} = \dfrac{4y+9}{y^2+5y+6}.$

13. (1) $\frac{3}{y} - \frac{3}{x} = -\frac{1}{4}$. (It should be noticed that these are simultaneous linear equations.)

(2) $\frac{6}{y} - \frac{3}{x} = \frac{1}{2}$.

Solution.

$\frac{3}{y} = \frac{3}{4}$, by subtracting equation (1) from (2).

$12 = 3y$, multiplying both sides by $4y$.

$y = 4$, dividing both sides by 3.

$\frac{3}{4} - \frac{3}{x} = -\frac{1}{4}$, substituting the value of y in (1).

$-\frac{3}{x} = -\frac{1}{4} - \frac{3}{4}$, subtracting $\frac{3}{4}$ from both sides.

$12 = x + 3x$, multiplying both sides by $-4x$.

$12 = 4x$, collecting terms.

Therefore, $x = 3$, dividing both sides by 4.

Check.

$\frac{3}{4} - \frac{3}{3} = -\frac{1}{4}$, by substituting in (1).

$\frac{3}{4} - 1 = -\frac{1}{4}$.

$-\frac{1}{4} = -\frac{1}{4}$.

$\frac{6}{4} - \frac{3}{3} = \frac{1}{2}$, by substituting in (2).

$\frac{3}{2} - 1 = \frac{1}{2}$.

$\frac{1}{2} = \frac{1}{2}$.

14. (1) $\frac{3}{r} - \frac{4}{s} = 1$.

(2) $\frac{4}{r} + \frac{3}{s} = 2$.

15. (1) $\frac{9}{u} - 3v = 4$.

(2) $\frac{3}{u} + 2v = \frac{10}{3}$.

16. (1) $\frac{1}{x} + \frac{1}{y} = 1$.

(2) $\frac{1}{x} - \frac{1}{y} = 3$.

17. (1) $\frac{7}{x-y} - \frac{8}{y-5} = 0$.

(2) $\frac{9}{2x-1} - \frac{5}{3y+4} = 0$.

18. $\dfrac{33}{x+5} + \dfrac{3.75}{.5(x-8.5)} = 0.$ **19.** $\dfrac{8v-7}{4v} + \dfrac{7}{2} = \dfrac{\frac{1}{2}-v}{v}.$

20. (1) $3y - \dfrac{5}{z} = 10.$ (2) $2y + \dfrac{10}{z} = 4.$

(b) Literal Linear Equations

160. A Literal Equation is an equation in which some, if not all, of the known numbers are represented by letters. Thus, $ax - b = cx - d$; and $\dfrac{a}{x} - b = c - \dfrac{d}{x}$, are literal equations; also, $ax - b = 2$ is literal.

161. Illustrative Problems.

1. Solve $\dfrac{a}{x} - b = c - \dfrac{d}{x}$

Solution. $a - bx = cx - d$, multiplying both sides by x.

$-bx - cx = -d - a$, subtracting a and cx from both sides.

$-(b + c)x = -(d + a)$, collecting terms.

Therefore, $x = \dfrac{d+a}{b+c}$, dividing both sides by $-(b + c)$.

2. The sum of two numbers is s and their difference is d. What are the numbers?

Solution.

Let x = the smaller of the two numbers.

Then $d + x$ = the greater number.

And $x + d + x = s$.

$x + x = s - d$, subtracting d from both sides.

$2x = s - d$, combining terms.

Therefore, $x = \dfrac{s-d}{2}$, dividing both sides by 2,

and $x + d = \dfrac{s-d}{2} + d = \dfrac{s+d}{2}$ = the greater number.

3. Solve for both unknown numbers:

(1) $\dfrac{a}{x}+\dfrac{b}{y}=c.$

(2) $\dfrac{a'}{x}+\dfrac{b'}{y}=c'.$

Solution.

(3) $\dfrac{ab'}{x}+\dfrac{bb'}{y}=b'c,$ equation (1) multiplied by b'.

(4) $\dfrac{a'b}{x}+\dfrac{bb'}{y}=bc',$ (2) multiplied by b.

(5) $(ab'-a'b)\dfrac{1}{x}=b'c-bc',$ subtracting (4) from (3).

(6) $ab'-a'b=(b'c-bc')x,$ multiplying both sides of (5) by x.

(7) Therefore, $x=\dfrac{ab'-a'b}{b'c-bc'},$ dividing both sides by $(b'c-bc')$.

(8) $\dfrac{aa'}{x}+\dfrac{a'b}{y}=a'c,$ equation (1) multiplied by a'.

(9) $\dfrac{aa'}{x}+\dfrac{ab'}{y}=ac',$ equation (2) multiplied by a.

(10) $(a'b-ab')\dfrac{1}{y}=a'c-ac',$ subtracting (9) from (8).

(11) $a'b-ab'=(a'c-ac')y,$ multiplying both sides of (10) by y.

(12) Therefore, $y=\dfrac{a'b-ab'}{a'c-ac'},$ dividing both sides by $(a'c-ac')$.

EXERCISE

162. In the following equations solve for x and y unless otherwise noted:

1. $ax + bx = 2(a^2 + ab).$

2. $ay + b = by + ab.$

3. $\frac{y}{a} + \frac{y}{b} + ab = bc + ac + \frac{y}{c}.$

4. (1) $x + y = a.$
   (2) $x - y = b.$ Solve for x and y in terms of a and b.

5. (1) $n + n' = s.$
   (2) $n - n' = d.$ Solve for n and n' in terms of s and d.

6. (1) $2e + k = a.$
   (2) $e - k = c.$ Solve for e and k.

7. (1) $\frac{m}{x} + \frac{n}{y} = f.$
   (2) $\frac{n}{x} - \frac{n}{y} = e.$

8. $\frac{b(x-b)}{x-a} + \frac{a(x-a)}{x-b} = a + b.$

9. (1) $kx - ex' = c.$
   (2) $k'x - e'x' = c'.$ Solve for x and x'.

10. $\frac{y}{mn} + \frac{y}{np} + \frac{y}{mp} = m + n + p.$ Find y in terms of m, n, and p.

11. (1) $ax + by = 2.$
    (2) $cx + dy = 2.$

12. (1) $a(x + y) = 10.$

(2) $c(x - y) = 4.$

13. (1) $\dfrac{x}{y} = a.$

(2) $x + y = b.$

14. $\dfrac{ar - b}{br} + \dfrac{br - a}{ar} = a + \dfrac{a - b}{abr}.$ Solve for r.

15. $\dfrac{a + y}{b + y} - \dfrac{a - y}{b - y} = \dfrac{2}{y^2 - b^2}.$

16. $gt^2 + vt = s.$ Solve for t. (Divide by g and complete the square.)

17. (1) $\dfrac{m + n}{m - n} = 15.$

(2) $7m - 8n = 0.$ Find m and n.

18. (1) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}.$

(2) $\dfrac{1}{x} - \dfrac{1}{y} = n.$

19. (1) $\dfrac{b}{x} + \dfrac{a}{y} = \dfrac{1}{5}.$

(2) $\dfrac{b}{x} - \dfrac{a}{y} = \dfrac{1}{7}.$ Solve for x and y.

20. (1) $\dfrac{1}{v'} - \dfrac{2}{v} = \dfrac{1}{a}.$

(2) $\dfrac{2}{v'} + \dfrac{1}{v} = \dfrac{2}{b}.$ Solve for v and v'.

21. $t^2 - 2t - (2m + 1) = 0.$ Solve for t.

22. $s^2 - (a - b)s - ab = 0.$

23. $(a + b)y^2 + (3a + b)y + 2a = 0.$

24. $\frac{n}{2}(2a + n - 1 + d) = s.$ Solve for n.

25. $\frac{ab}{x} = be + d + \frac{1}{x}\cdot$ Solve for x.

26. Solve each of the following for r:

$$(a)\ s = \frac{a}{l - r}.$$

$$(b)\ s = \frac{rl - a}{r - 1}.$$

27. Evaluate each of the expressions in the preceding example for s when $a = 2$, $l = 5$, and $r = \frac{1}{2}$.

In each of the following formulas, which are in fact literal equations, solve for the letter designated:

28. $C = \frac{E}{R + nr}\cdot$ Solve for r. **29.** $C = \frac{ne}{R + nr}\cdot$ Solve for n.

30. $a = \frac{n(a + 1)}{2}\cdot$ Solve for a. **31.** $\frac{V'}{V''} = \frac{P'}{P''}\cdot$ Solve for P''.

32. $\frac{1}{R} = \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''}\cdot$ Solve for R.

33. Solve the following for x and y:

(1) $x + ey = d.$
(2) $ax - y = f.$

34. (1) $\frac{b}{x} - \frac{c}{y} = 4.$

(2) $\frac{d}{x} + \frac{k}{y} = 5.$ Solve for x and y.

PROBLEMS

163. **1.** The sum of two numbers is s and their difference is d. What are the numbers?

2. Divide a dollars between two boys so that one shall have d dollars more than the other.

3. A line c inches in length is divided into two parts which are to each other as $m : n$. What is the length of each part?

4. Two men are d miles apart and are traveling towards each other, one at the rate of a miles an hour and the other at the rate of b miles an hour. In how many hours will they meet?

5. A boy in spending his money for oranges finds that if he had bought n more for the same money each would have cost him a cents less; if he had bought a fewer for the same amount each would have cost him b cents more. How many oranges did he buy?

Problems Yielding Fractional, Literal, Linear and Quadratic Equations.

6. A boy starts to walk from Oklahoma City to Norman, which is 18 miles distant, at the same time that another boy starts from Norman for Oklahoma City. If the first boy walks at the rate of 4 miles an hour while the second makes as many miles per hour as it requires hours for them to meet, where will they meet?

7. A tourist rode to the top of a mountain in an automobile at the rate of 8 miles an hour and walked down, after spending one hour and thirty minutes at the top, at the rate of 3 miles an hour. On his return he found that he had been gone 6 hours. How many miles was it to the top of the mountain?

8. A steamer running from New York City to Liverpool covers an average daily distance of 480 miles. It is followed after two days by a second steamer which makes an average of 660 miles

a day. In how many days will the second steamer overtake the first?

9. The difference between the third and fourth parts of a certain number exceeds the difference between the twelfth and sixteenth parts by 3. What is the number?

10. Find two consecutive numbers such that the fourth and eleventh parts of the less exceed the fifth and ninth parts of the greater by 1.

11. A father's age is equal to the combined ages of his two children and his wife. But in 26 years his age will be only half their combined ages. What is his age?

12. A chauffeur finds that he can make the round trip from Oklahoma City to Guthrie, traveling at the rate of 16 miles an hour, in a certain time; but that if he increases his rate on the outgoing trip 2 miles an hour and decreases his rate on the return trip by 1 mile an hour, it will save $\frac{1}{4}$ of an hour on the round trip. Find the distance between these two cities.

13. The product of two numbers added to their sum gives 19; and 5 times their sum taken from the sum of their squares leaves -10; find the numbers.

14. The diagonal of a rectangle is 9 inches longer than one of the sides and 63 inches longer than the other side. Find the area of the rectangle.

15. A train running at the rate of 60 miles an hour overtakes a man walking in the same direction and passes him in 10 seconds. It also meets another man walking in the opposite direction and passes him in 8 seconds. If both men walk at the same rate, find the length of the train.

16. In a certain election the successful candidate's vote was to that of his nearest competitor as 6 : 5, and the number of electors not voting was 1228. If the defeated candidate had received from those not voting an additional number of votes equal to $\frac{1}{10}$ of the entire vote of the precincts voting, he would have been elected by a majority of 200. What was the number of votes cast?

PROBLEMS FROM SCIENCE

164. 1. It is determined by laboratory experiment that the intensity of illumination from a light is directly proportional to the candle-power and inversely proportional to the square of its distance from the source.

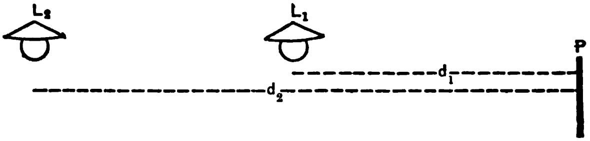

FIG. 53

In Fig. 53, let L_1 and L_2 denote the positions of two lamps, C_1 and C_2 their respective candle-powers and d_1 and d_2 their respective distances from P, a point in the same straight line with L_1 and L_2.

Denoting intensity of illumination by I, it follows that

$$(a)\ \frac{I_1}{I_2}=\frac{C_1}{C_2},\qquad (b)\ \frac{I_1}{I_2}=\frac{d_2^2}{d_1^2},\qquad (c)\ \frac{I_1^2}{I_2^2}=\frac{C_1\,d_2^2}{C_2\,d_1^2}.$$

It should be noted that the foregoing equations are linear in C but quadratic in d. Solve (c) for d_1.

2. As a vessel enters the harbor two signal lights having 2000 and 1500 candle-power respectively and situated a half mile apart appear to lie nearly in a straight line and to be equally bright. How far is the vessel from the nearer light?

(Suggestion: $I_1 = I_2$; solve (c) for d_1.)

3. A single arc lamp illuminates a screen as brightly as two such lamps of the same power placed 10 feet farther from the screen. What is the distance of the two lamps from the screen?

4. Assuming that the above equations hold also for heat, find the distance of a person from a fire if by moving 5 feet farther away the intensity of the heat is $\frac{1}{3}$ as great as before.

5. How far is a person from a lamp when by moving 3 feet nearer the intensity is 6 times as great as before?

6. The horse-power produced by water running over a dam is directly proportional to the height of the dam. At Niagara Falls the water falls 165 feet and the power thus generated is estimated at 1,000,000 horse-power. What will be the power produced when the fall has been reduced to 100 feet?

7. The most pleasing rectangle to the eye is that one having the ratio 7 : 5 between its length and width. What then should be the width of a door 7 feet 7 inches high?

8. The steepest part of the railway up Pike's Peak has a rise of 1 foot in 2 feet 7 inches of track. Were this rise uniform what would be the rise in 1 mile?

9. The law governing the revolution of the planets about the sun is stated thus:

$$\frac{P_1^2}{P_2^2} = \frac{d_1^3}{d_2^3},$$

where P_1 and P_2 denote the periods of revolution; d_1 and d_2 the mean distances from the sun. Assuming the earth's distance from the sun to be 92,000,000 miles, its period as 365 days and the period of Venus as 225 days, find the distance of Venus from the sun.

10. Assuming 687 days as Mars's period and the other conditions as in the preceding problem, find Mars's distance from the sun.

11. A train running at a uniform rate required 2 hours more time to make a trip than would have been necessary if its rate had been 8 miles an hour more. But if the rate had been 10 miles an hour less, it would have required 4 hours more to make the same trip. What was the length of the trip?

12. An automobile can travel over a certain course in one hour less time by increasing its speed 3 miles an hour; it can also cover the same course in $1\frac{1}{2}$ hours less time by increasing its speed 5 miles an hour. What is the length of the course?

13. Two persons 10 miles apart, starting at the same time, will come together in 6 hours if they both walk in the same direction, or in 2 hours if they walk towards each other. At what rates are they walking?

14. In placing telegraph poles between two cities, it was found that if the poles had been placed 12 feet farther apart, 4 poles less to the mile would have been required. How far apart were the poles placed?

15. In copying a manuscript containing 12,000 words, a boy finds that he can write twenty-five words more per minute by using a typewriter than by using a pen. If the typewriter saves him 1 hour and 20 minutes in copying the manuscript, what was his rate per minute with the typewriter?

16. The hypotenuse of a right triangle is 20 and one of the other two sides is 12. Find the length of the other side. (Suggestion: Let $x =$ length of one side.)

17. The area of a triangle is 40 square inches and its base is 2 inches longer than its altitude. What is the altitude?

18. A tree standing on level ground was broken over so that the top reached the ground 50 feet from the foot of the tree, the other end of the part broken off remaining attached to the stump. The stump was 25 feet more than $\frac{1}{3}$ the height of the tree. What was the height of the tree?

19. For a yacht race a 40-mile course was marked out in the form of a right triangle one leg of which was 10 miles longer than the other. What was the length of each leg of the course?

(c) Linear Systems

164. Systems Involving More than Two Unknowns.

Given the equations:

(1) $x + y + z = 6,$

(2) $2x + 3y + 4z = 22,$

(3) $2x + y + z = 7,$

we may obtain:

(4) $3x + 4y + 5z = 28$, by adding (1) and (2),
(5) $5x + 5y + 6z = 35$, by adding (1), (2) and (3),
(6) $x + 2y + 3z = 16$, by subtracting (1) from (2).

Equations (1), (2), and (3) are *independent* with respect to each other, since none of them can be obtained from the others. Equations (4), (5), and (6) are called *derived* equations with respect to equations (1), (2), and (3), since each of them was obtained by adding or subtracting (1), (2), and (3).

An equation in three or more letters is linear if no one of the letters is multiplied by itself and no two of the letters occur in the same term as a product. Equations (1), (2), and (3) are linear.

A system of three or more independent linear equations containing three or more unknown numbers has *one set* of roots and but one, provided that no two of the given equations are inconsistent.

166. Solution of a System of Three Linear Equations.

Example 1. Solve for x, y and z the following system:

(1) $x + y + z = 6.$
(2) $x + 2y + z = 8.$
(3) $2x - y - z = 3.$

Solution.

(4) $y = 2$, subtracting (1) from (2).
(5) $3x = 9$, adding (1) and (3).
(6) $x = 3$, dividing both sides of (5) by 3.
(7) $3 + 2 + z = 6$, substituting values of x and y in (1).

(8) $z = 6 - 3 - 2$, subtracting 3 and 2 from both sides of (7).

(9) $z = 1$, combining terms in 2d member of (8).

Therefore, $x = 3$, $y = 2$ and $z = 1$ is the required set of roots.

Check. $3 + 2 + 1 = 6$, substituting values of x, y and z in (1).

$6 = 6$.

$3 + 2\ (2) + 1 = 8$, substituting values of x, y and z in (2).

$8 = 8$.

$2\ (3) - 2 - 1 = 3$, substituting values of x, y and z in (3).

$3 = 3$.

Example 2. Solve for x, y and z, the following system:

(1) $x + ay = b$.
(2) $ax + z = c$.
(3) $z + cy = a$.

Solution.

(4) $ax + a^2y = ab$, multiplying (1) by a so as to eliminate x between (1) and (2).

(2) $ax + z = c$.

(5) $z - a^2y = c - ab$, subtracting (4) from (2).

(3) $z + cy = a$.

(6) $cy + a^2y = a - c + ab$, subtracting (5) from (3).

(7) $(c + a^2)y = a - c + ab$, collecting terms in (6).

(8) $y = \dfrac{a - c + ab}{c + a^2}$, dividing both sides of (7) by $(c + a^2)$.

(9) $x = b - ay$, subtracting ay from both sides of (1).

(10) $x = b - \dfrac{a(a - c + ab)}{c + a^2}$, substituting value of y in (9).

(11) $x = \dfrac{a^2b + bc - a^2 + ac - a^2b}{c + a^2}$, simplifying (10).

(12) $x = \dfrac{bc - a^2 + ac}{c + a^2}$, simplifying (10).

(13) $z = a - cy$, subtracting cy from both sides of (3).

(14) $z = a - \dfrac{c(a - c + ab)}{c + a^2}$, substituting value of y in (13).

(15) $z = \dfrac{ac + a^3 - ac + c^2 - abc}{c + a^2}$, simplifying (14).

(16) $z = \dfrac{a^3 + c^2 - abc}{c + a^2}$, combining similar terms.

Therefore,

$$y = \frac{a - c + ab}{c + a^2}, \quad x = \frac{bc - a^2 + ab}{c + a^2} \quad \text{and} \quad z = \frac{a^3 + c^2 - abc}{c + a^2}$$

are the required roots.

From the foregoing examples it will be seen that the solution of a system of three linear equations in three unknowns may be obtained as follows:

(1) *Eliminate one unknown from any pair of the given equations and the same unknown from any other pair, thereby deriving from the given system of three equations with three unknowns a new system of two equations with two unknowns.*

(2) *This new system may be solved by methods already given.*

(3) *Having found the values of two of the unknowns the third may then be found by substituting in one of the given equations that contains the third unknown, the values of the other two, and solving the resulting equation which now contains but one unknown.*

Systems consisting of four or more equations with four or more unknowns may be solved in a similar manner.

EXERCISE

Solve the following systems and check results:

1. (1) $x + y + z = -1.$
(2) $5x + 3y + 2z = 1.$
(3) $3x - y - 5z = 13.$

2. (1) $y + 8x + 5z = 1.$
(2) $3y + 4x + 10z = -5.$
(3) $x = 4z.$

3. (1) $x + \frac{1}{2}y = 55.$
(2) $\frac{1}{2}x + y = 50.$
(3) $z + \frac{1}{2}y = 50.$

4. (1) $x_1 + x_2 - x_3 = 5.$
(2) $x_1 - x_2 - x_3 = 1.$
(3) $x_1 + 2x_2 - 3x_3 = -3.$

5. (1) $5y + 3x = 65.$
(2) $2x - z = 11.$
(3) $3y + 4z = 57.$

6. (1) $3r + s + t = 9.$
(2) $r - s + 2t = 6.$
(3) $r + 2s - t = 0.$

7. (1) $ax + by = c.$
(2) $x + dz = e.$
(3) $cy + z = f.$ Solve for x, y, and z.

8. (1) $x + y + z = 0.$
(2) $5x + 3y + 4z = -1.$
(3) $2x - 7y + 6z = 21.$

9. (1) $ax - by = 1.$
(2) $cx + dy = 2.$
(3) $ay + bz = 3.$

Solve for x, y, and z.

10. (1) $\frac{6}{x} + \frac{12}{y} + \frac{18}{z} = 18.$
(2) $\frac{12}{x} + \frac{18}{y} + \frac{6}{z} = 23.$
(3) $\frac{18}{x} + \frac{6}{y} + \frac{12}{z} = 25.$

PROBLEMS

167. **1.** It is known from geometry that the tangents to a circle from a point without are equal. Hence, when a circle is inscribed in a triangle, as in Fig. 55, the tangents *Cm* and *Cn* are equal; also *Am* and *Ap* are equal and *Bp* and *Bn* are likewise equal.

If $a = CB = 60$, $b = CA = 100$, and $c = AB = 90$, how long are the tangents?

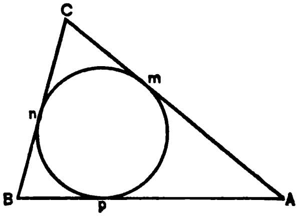

FIG. 55

2. In triangle *ABC*, Fig. 56, $a + b = 48$, $b + c = 41$ and $a + c = 43$, where a, b and c denote the three sides. Find the length of each side.

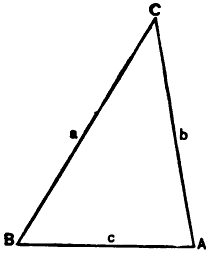

FIG. 56

3. The perimeter of a certain triangle is 60 feet. Two of its sides are equal and the third is 10 feet shorter than the other two together. What is the length of each side?

4. Two of the angles of a triangle are equal and their sum is $\frac{8}{5}$ of the third angle. How many degrees in each angle?

5. Angle *A* of a certain triangle is 17° larger than angle *B* and angle *B* is 23° larger than angle *C*. How many degrees in each angle?

6. Diophantus, the Grecian mathematician, who lived in Alexandria in the first half of the fourth century, gives this problem: "Find three numbers such that the sum of each pair is a given number." Supposing the given numbers are 20, 30 and 40, solve this system.

7. Suppose the given numbers of the preceding problem are a, b and c. Solve the system

(1) $x + y = a,$
(2) $x + z = b,$
(3) $y + z = c,$ for x, y and z.

8. The sum of three numbers is 70. One-half the first and second plus the third equals the first number; and twice the third plus the first equals three times the second.

9. In a certain triangle angle A exceeds angle B by $10°$ and angle B exceeds angle C by $20°$. How many degrees in each angle?

10. The sum of the dimensions of a rectangular box is 48 inches. The breadth equals one-half the sum of the length and depth; and the depth is $\frac{1}{3}$ of the sum of the length and breadth. What are the dimensions of the box?

CHAPTER XI

SIMULTANEOUS QUADRATICS

168. Systems Involving Quadratics. The number of systems of equations of the second degree in two variables that are solvable by quadratics is *limited.* The algebraic solution of such systems in general is beyond the scope of a first course in algebra. Three types of these systems, however, will be considered in this chapter.

169. Linear and Quadratic. The solution of two simultaneous equations, one being linear and the other quadratic, is obtained by the method of substitution.

Example 1.

(1) $x - y = 5.$

(2) $xy = 24$, solve for x and y. (Equation (2) is of the second degree. Why?)

Solution. $x = 5 + y$, equation (1) solved for x in terms of y.

$y(5 + y) = 24$, substituting the value of x in equation (2).

$y^2 + 5y = 24$, removing the parentheses.

$y^2 + 5y - 24 = 0$, subtracting 24 from both sides.

$(y - 3)(y + 8) = 0$, factoring the last equation.

$y - 3 = 0$, or $y + 8 = 0$, equating factors to zero.

Therefore, $y = 3$ or $y = -8$.

And since, $x = 5 + y$,

$x = 5 + 3$, or $x = 5 - 8$.

Therefore, $x = 8$, or $x = -3$.*

Check. $8 - 3 = 5$, substituting for x and y, in equation (1).

also, $-3 - (-8) = 5$

$5 = 5$.

$3 \cdot 8 = 24$, substituting for x and y in equation (2).

$(-3)(-8) = 24$, substituting for x and y in equation (2).

$24 = 24$.

Example 2.

(1) $x^2 + 3y^2 = 31$.

(2) $x + y = 5$, solve for x and y.

Solution.

$x = 5 - y$, x in terms of y from equation (2).

$(5 - y)^2 + 3y^2 = 31$, substituting for x in equation (1).

$25 - 10y + y^2 + 3y^2 = 31$, removing parentheses.

$y^2 + 3y^2 - 10y = 31 - 25$, subtracting 25 from both sides.

$4y^2 - 10y = 6$, combining similar terms.

$2y^2 - 5y = 3$, simplifying by dividing both sides by 2.

$2y^2 - 5y - 3 = 0$, subtracting 3 from both sides.

$(2y + 1)(y - 3) = 0$, factoring.

Hence, $y = -\frac{1}{2}$, or, $y = 3$, equating factors to zero and solving each.

Since, $x = 5 - y$, $x = 5 - 3 = 2$, when $y = 3$.

$x = 5 - (-\frac{1}{2}) = 5\frac{1}{2}$, when $y = -\frac{1}{2}$.

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* Observe that $x = 8$ when $y = 3$; and $x = -3$ when $y = -8$.

EXERCISE

Find two values for each letter representing an unknown number in the following:

1. $xy = 12.$ $\frac{x}{y} = 3.$

2. $x + y = 8.$ $xy = 12.$

3. $xy = 15.$ $x - y = 2.$

4. $xy = 6.$ $3x - 2y = 16.$

5. $u + v = -10.$ $u^2 + v^2 = 122.$

6. $s - t = 4.$ $2st = 42.$

7. $3gh = 10.$ $36g + 36h = 289.$

8. $r^2 + s^2 = 45.$ $r - s = 3.$

9. $x' - x'' = 36.$ $x' \cdot x'' = 37.$

10. $x' + x'' = 9.$ $(x')^2 + (x'')^2 = 27.$

11. $v = gt.$ $s = vt + \frac{1}{2}gt^2$ find v and t.

12. $e^2 + 9t + 4e = 68.$ $2e + 3t = 2$, solve for t and e.

13. $3r + 2s = 5.$ $rs - 6r = -3.$

14. $v = \sqrt{2gs}.$ $s = vt + \frac{1}{2}gt^2$, solve for v and t.

15. $h^2 + k^2 + k = 10.$ $h + k + 1 = 0.$

PROBLEMS

170. 1. A box is to be made having a capacity of 96 cubic feet. Its length is to be 4 times its depth and its width is to be 3 times its depth. What will be its dimensions?

2. The perimeter of a rectangle is 80 feet and its area is 300 square feet. Find its dimensions.

3. A lot having an area of 5000 square feet is surrounded by a walk 5 ft. wide. A fence outside the walk, and extending around the whole lot, is 340 feet long. What are the dimensions of the lot?

4. The sum of two numbers is 5 and the square of the first plus twice the square of the second is 22. What are the numbers?

5. The product of two numbers is 12 and their quotient is 3. What are the numbers?

171. Two Quadratics of the type $ax^2 + by^2 + c = 0$. The solution of two quadratic equations, each of which is of the form $ax^2 + by^2 = c$, is obtained by eliminating one of the unknowns by addition or subtraction.

Example. $3x^2 - 5y^2 = 3$.

$x^2 + y^2 = 25$, solve for x and y.

Solution.

$3x^2 - 5y^2 = 3$, equation (1) as given.

$3x^2 + 3y^2 = 75$, equation (2) multiplied by 3.

$8y^2 = 72$, equation (1) subtracted from equation (2), multiplied by 3.

$y^2 = 9$, dividing both sides by 8.

$y = \pm 3$, taking square roots of both sides.

Since, $x = \pm\sqrt{25 - y^2}$, equation (2) solved for x

$= \pm\sqrt{25 - 9}$, substituting the value of y

$= \pm\sqrt{16}$, combining terms

$= \pm 4$, taking square roots of both sides.

Check. $3(4^2) - 5(3^2) = 3$, substituting values of x and y in equation (1).

$48 - 45 = 3$, simplifying.

$3 = 3$.

$4^2 + 3^2 = 25$, substituting in equation (2).

$25 = 25$.

EXERCISE

Solve for both unknown numbers:

1. $x^2 + y^2 = 41.$
$x^2 - y^2 = 9.$

2. $4s^2 + 7t^2 = 32.$
$3s^2 - 11t^2 = -41.$

3. $n^2 + m^2 = 13.$
$5n^2 - 2m^2 = 2.$

4. The sum of the squares of two numbers is 194, and the difference of their squares is 144. What are the numbers?

5. The square of the hypotenuse of a right triangle is 100 and the difference in the squares of the other two sides is 28. Find the sides of the triangle.

172. Homogeneous Quadratics. An equation is homogeneous when all of its terms containing the unknown numbers are of the same degree.

Thus, $ax^2 + bxy + cy^2 = 5$ is a homogeneous quadratic in x and y, since each term containing the unknowns is of the second degree.

Also, $ax^3 + bx^2y + cxy^2 + dy^3 = e$ is a homogeneous equation of the third degree.

A system consisting of two homogeneous quadratics may be solved in either of two ways as illustrated by the following example:

Example. (1) $x^2 - xy = 54.$
(2) $xy - y^2 = 18.$

First solution.

(1) $x^2 - xy = 54.$

(3) $3xy - 3y^2 = 54$, (2) multiplied by 3.

(4) $x^2 - 4xy + 3y^2 = 0$, subtracting (3) from (1).

(5) $(x - 3y)(x - y) = 0$, factoring (4).

(6) $x = 3y$ and $x = y$, equating factors of (5) to zero, etc.,

(7) $9y^2 - 3y^2 = 54$, substituting $x = 3y$ in (1).

(8) $6y^2 = 54$, combining terms in (7).

(9) $y = \pm 3$, solving (8) for y.

(10) $x = 3(\pm 3) = \pm 9$, substituting $y = \pm 3$ in (6).

(11) $y^2 - y^2 = 54$, substituting $x = y$ in (1).

(12) $0 = 54$, which is impossible.

Hence, the solution is $x = \pm 9$, $y = \pm 3$.

Check. $(\pm 9)^2 - (\pm 9)(\pm 3) = 54$, substituting in (1).

$$81 - 27 = 54.$$

$$54 = 54.$$

And $(\pm 9)(\pm 3) - (\pm 3)^2 = 18$, substituting in (2).

$$27 - 9 = 18.$$

$$18 = 18.$$

Example. (1) $x^2 - xy = 54$.

(2) $xy - y^2 = 18$, same system as before.

Second solution.

(3) $x^2 - vx^2 = 54$, putting $y = vx$ in (1).

(4) $vx^2 - v^2x^2 = 18$, putting $y = vx$ in (2).

(5) $x^2(1 - v) = 54$, collecting x^2 terms in (3).

(6) $x^2(v - v^2) = 18$, collecting x^2 terms in (4).

(7) $x^2 = \dfrac{54}{1 - v}$, solving (5) for x^2.

(8 $x^2 = \dfrac{18}{v - v^2}$, solving (6) for x^2.

(9) $\dfrac{54}{1 - v} = \dfrac{18}{v - v^2}$, equating values of x^2.

(10) $54\,v = 18$, multiplying both members of (9) by $v - v^2$.

(11) $v = \dfrac{1}{3}$, solving (10) for v.

(12) $y = \dfrac{x}{3}$, since we assumed $y = vx$.

(13) $x^2 - \dfrac{x^2}{3} = 54$, substituting $y = \dfrac{x}{3}$ in (1).

(14) $x = \pm 9$, solving (13) for x.

(15) $y = \pm 3$, since $y = \dfrac{x}{3}$ in (1).

Hence the solution is $x = \pm 9$, $y = \pm 3$.

Check. These roots, being the same as already found and checked, require no further checking.

173. Summary. To solve a system consisting of two homogeneous quadratics in x and y.

First Method:

(1) *Eliminate the constant terms by addition or subtraction of the two equations after first multiplying by such number or numbers as will make these terms equal.*

(2) *Factor the equation thus derived every term of which contains one or both of the unknowns and by equating these factors separately to zero determine the value of x in terms of y.*

(3) *Substitute this value for x in one of the original equations and solve the resulting quadratic in y in the usual way.*

(4) *Determine x from the known value of x in terms of y.*

(5) *Check all solutions by substituting in both equations of the system.*

Second Method:

(1) *Assume* $y = vx$ *and substitute this value of* y *in both equations of the system.*

(2) *Solve each of the resulting equations for* x^2 *and equate the two values thus obtained. This will give an equation in* v.

(3) *Solve for* v *and substitute in* $y = vx$.

(4) *Substitute the value of* y *in terms of* x *in either equation of the given system and complete the work as by the first method.*

It will readily be seen that the difference in the foregoing solutions is in the method by which the value of x in terms of y is determined.

EXERCISE

174. Solve the following systems and check solutions:

1. $x^2 + y^2 = 13$.
$xy = 6$.

2. $x^2 + xy - 2y^2 = 4$.
$x^2 - 3xy = -2$.

3. $x^2 + xy = 12$.
$y^2 - 24 = -xy$.

4. $2x^2 - xy = 28$.
$x^2 + 2y^2 = 18$.

5. $5x^2 - y^2 = 1$.
$3y^2 - xy - 10 = 0$.

6. $s^2 + st - 5t^2 = 25$.
$s^2 + 4t^2 - 40 = 0$.

7. $x^2 - xy + y^2 - 21 = 0$.
$y^2 - 2xy + 15 = 0$.

8. $\frac{a}{x^2} + \frac{b}{y^2} = 0$.
$2xy = C$.

(Suggestion: In 8 determine x in terms of y from equation (1)).

9. (1) $a^2r^2 + b^2s^2 = C^2$.
(2) $r^2 - 2rs + s^2 = c^2d^2$. Solve for r and s.

(Suggestion: $r = s + cd$ or $r = s - cd$ by factoring (2), equating factors to zero.)

10. $3x^2 + 3xy = 36.$
$2xy + 2y^2 = 48.$

11. Solve systems in 1 – 7 by both methods.

175. Symmetric Equations of the third or fourth degree may often be solved readily by dividing the variable member of one equation by the variable member of the other.

Example 1. (1) $x^3 + y^3 = 35.$
(2) $x + y = 5.$

Solution.

(3) $x^2 - xy + y^2 = 7$, (1) divided by (2).

(4) $x = 5 - y$, (2) solved for x in terms of y.

(5) $(5 - y)^2 - y(5 - y) + y^2 = 7$, substituting for x in (3).

(6) $3y^2 - 15y + 18 = 0$, (5) simplified.

(7) $(3y - 6)(y - 3) = 0$, (6) factored.

(8) Therefore, $y = 2$ or 3, equating factors of (7) to zero, etc.,

(9) and $x = 3$ or 2, substituting $y = 2$ or 3 in (4).

Check. $(3)^3 + (2)^3 = 35$, substituting $x = 3$ and $y = 2$ in (1).
$27 + 8 = 35.$
$35 = 35.$
$3 + 2 = 5$, substituting $x = 3, y = 2$ in (2).
$5 = 5.$
$(2)^3 + (3)^3 = 35$, substituting $x = 2, y = 3$ in (1).
$35 = 35.$
$2 + 3 = 5$, substituting $x = 2, y = 3$ in (2).
$5 = 5.$

Example 2. (1) $x^4 + x^2y^2 + y^4 = 21.$

(2) $x^2 + xy + y^2 = 7.$

Solution.

(3) $x^2 + y^2 - xy = 3$, dividing (1) by (2).

(4) $x^2 + y^2 = 5$, adding (2) and (3) and simplifying.

(5) $2xy = 4$, subtracting (3) from (2).

(6) $x + y = 3$, adding (5) to (4) and taking square roots.

(7) $x - y = 1$, subtracting (5) from (4) and taking square roots.

From (6) and (7) it is readily seen that $x = 2$ and $y = 1$ are the required roots.

EXERCISE

Solve the following systems as illustrated in the foregoing:

1. $x^3 + y^3 = 91.$
   $x + y = 7.$

2. $x^2 + xy + y^2 = 7.$
   $x^4 + x^2y^2 + y^4 = 91.$

3. $x^3 + y^3 = b.$
   $x + y = a.$

4. $1 + y^3 = x^3,$
   $1 + y = x.$

5. $x^2 + xy = \frac{5}{12}.$
   $y^2 + xy = \frac{5}{18}.$

6. $x^3 + y^3 = 2a^3 + 6a.$
   $x^2 - xy + y^2 = a^2 + 3.$

7. $x^2 + xy + y^2 = 21.$
   $x + \sqrt{xy} + y = 7.$

8. $x^3 + y^3 - 218 = 0.$
   $x^2 - xy + y^2 = 109.$

9. $x + y = 65.$
   $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5.$

10. $\sqrt[3]{x} + \sqrt[3]{y} = 4.$
    $x + y = 28.$

PROBLEMS

176. 1. The hypotenuse of a right triangle is 20 feet and the sum of the other two sides is 28 feet. What is the length of the sides?

2. Find two numbers such that their sum is a and their product b.

3. The difference between two numbers is 2 and the difference between their cubes is 98. Find the numbers.

4. The sum of the sides of two squares is 104 feet and the sum of their areas is 346 square feet. What are the sides?

5. The hypotenuse of a right triangle exceeds one of its sides by 2 inches and the other by 4 inches. What are the sides?

6. The mean proportional between two numbers is 6 and the sum of their squares is 97. Find the numbers.

7. The perimeter of a rectangle is a feet and its area is b square feet. What are its dimensions?

8. The sides of a triangle are 4, 5 and 6. Find the altitude considering 5 as the base.

(Suggestion: Represent segments of base on either side of the foot of the perpendicular by x and y.)

9. The dimensions of a rectangle are in the ratio of 3 to 5 and the length of its diagonal is $2\sqrt{34}$. Find the dimensions.

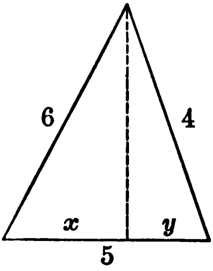

FIG. 57

10. The legs of a right triangle are 3 and 4. A line is drawn from the vertex of the right angle to a point in the hypotenuse dividing the given triangle into two triangles (a) which have equal perimeters, (b) equal areas.

Where does the line meet the hypotenuse in (a)? In (b)?

PROBLEMS FOR REVIEW OF QUADRATICS

177. 1. A ladder 25 feet long and standing in a street will just reach a window on one side of the street 20 feet above ground; if turned over on its foot to the other side of the street, it will just reach a window 15 feet above ground. Find the width of the street.

2. In a right triangle one leg is 5 feet longer than the other; while the hypotenuse, c, is 5 feet longer than the other leg. What are the lengths of the sides? (Fig. 58.)

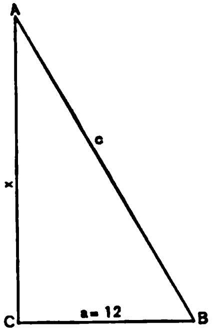

Fig. 58

3. The sum of the squares of two numbers is 41; the difference between the numbers is $\frac{1}{9}$ of their sum. Find the numbers.

4. A number added to its reciprocal gives $\frac{25}{12}$. What is the number?

5. Upon being wet cloth shrinks $\frac{1}{8}$ in length and $\frac{1}{16}$ in width. If the number of yards of cloth in a certain piece is decreased by $5\frac{3}{4}$, by being wet, and the perimeter of the piece after shrinking is 38 yards, what were the dimensions of the piece before being wet?

6. Find three consecutive numbers having a product equal to 3 times the middle number.

7. A park is 80 rods long and contains 60 acres. Two driveways of the same width cross the park at right angles to each other and parallel to the sides. If the area of the two driveways together is 784 square rods, what is their width?

8. A farmer is cutting wheat in a field 80 rods long and 60 rods wide. How wide a strip must he cut around the outside of the field in order that $\frac{1}{2}$ of the field may be left uncut?

9. A bin is to be built to hold 10 tons of coal. It is to be

twice as long as it is wide and 5 feet deep. Allowing 40 cubic feet to the ton, what will the other dimensions of the bin be?

10. The area of a rectangle is 120 square feet and the length of its diagonal is 17 feet. What are its dimensions?

PROBLEMS FROM SCIENCE

178. $s = \frac{1}{2}gt^2$, in which t denotes seconds of time, $g = 32.16$, and s distance in feet, is an equation giving the relation between these quantities; either t or s being known, the other may be found from the foregoing equation.

$s = vt + \frac{1}{2}gt^2$, and $s = vt - \frac{1}{2}gt^2$, in which v denotes initial velocity and the other letters have the same meaning as before, give the distances s passed through in t seconds by a body projected vertically upward or downward.

1. A ball was dropped from the top of a high building and reached the ground in 7 seconds. How high was the building?

2. An arrow was shot vertically upward and struck the top of a steeple in 5 seconds. If the velocity of the arrow at the start was 600 feet per second, what was the height of the steeple?

3. A ball was thrown downward from the top of a high cliff with an initial velocity of 300 feet per second, and struck the ground in 7 seconds. How high was the cliff?

4. A body is projected upward with a velocity of 200 feet per second. When will it be at a height of 100 feet?

5. A ball is thrown horizontally from the roof of a high building with a velocity of 100 feet per second. Supposing that it will strike the ground in the same time as if allowed to fall vertically and that it reaches the ground in 5 seconds, what is the height of the building?

6. $v = \sqrt{2gs}$, gives the velocity that a body must have in order to rise to a height of s feet. What must be the initial velocity of a bullet, that is, what must be its velocity on leaving the gun, if it is to pass over the Washington monument, height 555 feet?

7. The value of g on the moon is 5.4; compare the distances passed over in 5 seconds by bodies falling freely on the moon and the earth.

8. From the equation $v = \sqrt{2gs}$ determine the height to which a ball fired upward with an initial velocity of 2200 feet per second will rise, (*a*) if fired from the earth; (*b*) if fired from the moon?

9. To determine his height a balloonist drops a torpedo which will explode on striking the ground, and observes that it is 12 seconds before the explosion takes place. How high was he?

10. Experience has demonstrated that the best speed is attained by making the driving wheels of a locomotive 6 feet in diameter. What is the diameter of the smaller carrying wheels, if they make 120 more revolutions in going a mile than do the drivers? (Mile = 5280 feet.)

CHAPTER XII

THE BINOMIAL FORMULA—FACTOR THEOREM

179. The Binomial Theorem for positive integral exponents. We have already learned that

$$(a + b)^2 = a^2 + 2\,ab + b^2,$$

and $$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$

$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$

as may be shown by actual multiplication. Suppose, now, that n represents the exponent in any one of the above cases, it will be observed that

(1) The first term in each result is a^n.
(2) The second term in each is na^{n-1}.
(3) The exponents of a decrease by unity from term to term while the exponents of b increase by unity from term to term.
(4) The coefficient of any term after the first is equal to the coefficient of the next preceding term multiplied by the exponent of a in that term and this product divided by the exponent of b increased by unity.
(5) The number of terms in the result is $n + 1$. The above process is called the expansion of a binomial. These results hold true for $n = 2$, $n = 3$, and $n = 4$, as shown in the examples given, and it can be proven that results obtained

in this way will hold for any integral value of n. Hence, we have the following formula, known as the binomial formula:

$$\text{XXXI} \quad (a+b)^n = a^n + na^{n-1} \cdot b + \frac{n(n-1)a^{n-2} \cdot b^2}{\underline{|2}^*} + \frac{n(n-1)(n-2)a^{n-3}b^3}{\underline{|3}}$$

$$+ \dots \frac{n(n-1)\dots(n-r+2)}{\underline{|(r-1)}} a^{n-r+1} \cdot b^{r-1} + \dots + b^n.$$

The expression $\dfrac{n(n-1)\dots(n-r+2)}{\underline{|(r-1)}} a^{n-r+1} \cdot b^{r-1}$ is known as the rth term of the expansion of the binomial. By means of this expression any desired term of the result may be written down at once by substituting the proper values for n and r.

For example, it is desired to know the 7th term of the expansion of $(a + b)^8$.

Here $n = 8$, $r = 7$, $r - 1 = 6$, $n - r + 2 = 3$, and $n - r + 1 = 2$. This gives

$$\frac{\overset{4}{\cancel{8}} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3}}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \times a^2b^6 = 28\, a^2b^6.$$

ORAL EXERCISE

180. Express as a power of a binomial:

1. $x^3 - 3x^2y + 3xy^2 - y^3$. 2. $-x^3 + 3x^2 - 3x + 1$.

3. $1 - 3a + 3a^2 - a^3$. 4. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

---

* $\underline{|2}$ is read factorial two and means 2×1. $\underline{|3} = 3 \times 2 \times 1$ and $\underline{|4} = 4 \times 3 \times 2 \times 1$. In general $\underline{|n} = n(n-1)(n-2)(n-3)\dots[n-(n-1)]$.

5. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$. 6. $8c^3 - 12c^2d + 6cd^2 - d^3$.

7. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

8. $16a^4 - 32a^3 + 24a^2 - 8a + 1$.

9. $t^3 + 3t^2u + 3tu^2 + u^3$. 10. $a^{3p} + 3a^{2p}b^q + 3a^pb^{2q} + b^{3q}$.

EXERCISE

181. Expand by formula XXXI:

1. $(1 - a)^3$.
2. $(2a - b)^4$.
3. $(2x - 3y)^3$.
4. $(ax + by)^5$.
5. $(x^p - y^q)^6$.
6. $(m - 2n)^4$.
7. $(x^2 - z^3)^5$.
8. $\left(\frac{a}{b} - cd\right)^3$.
9. $[a - (b + c)]^4$.
10. $[(a - b) + (c - d)]^6$.

Give the first three and the last terms of each of the following:

11. $(p^a + q^c)^{10}$.
12. $\left(\frac{a}{w} - \frac{b}{a}\right)^{15}$.
13. $(2 - c^2)^8$.
14. $\left(\frac{1}{3} + \frac{a}{b}\right)^9$.

Using the expression for the rth term determine:

15. The 5th term of $(a - b)^{10}$.
16. The 7th term of $(1 - a)^{10}$.
17. The 10th term of $(2a + 3b)^{12}$.
18. The 9th term of $(p + q)^9$.
19. The 4th term of $\left(a - \frac{b}{2}\right)^5$.
20. The 15th term of $(x - 1)^{20}$.

21. The 7th term of $\left(\frac{p}{q} - \frac{c}{d}\right)^8$. **22.** The 3d term of $\left(\frac{a}{2} + \frac{b}{3}\right)^5$.

23. The 3d term of $[a + (b - c)]^5$.

24. The 11th term of $[b - (c + d)]^{12}$.

25. The 4th term of $(a^{-x} + b^{-y})^6$.

182. Factor Theorem. $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \dots a_n$, in which a_0, a_1, etc., are any real numbers, is the general form of a polynomial in x. Equated to zero this gives

$$(A) \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

as the general equation of the nth degree in x.

Factor Theorem: If b is a root of the equation (*A*) *then $x - b$ is a factor of its left-hand member.*

Thus, $x^3 - 3x^2 + 2 = 0$ is satisfied for $x = 1$, as may be shown by substituting 1 for x. Hence $x - 1$ is a factor of $x^3 - 3x^2 + 2$.

Again, $x^n - b^n = 0$ is satisfied for $x = b$, since by substitution we have $b^n - b^n = 0$ or $0 = 0$. Hence, $x^n - b^n$ is exactly divisible by $x - b$, which is therefore a factor of the polynomial.

The factor theroem is useful in determining the binomial divisors of any polynomial; also in determining roots of equations.

EXERCISE

Using the factor theorem, determine a binomial factor of each of the following:

1. $a^7 + b^7$.
2. $x^6 - y^6$.
3. $1 - x^5$.
4. $a^n - b^n$.

5. $x^3 + 3x^2y + 3xy^2 + y^3$. 6. $t^3 - 3t^2u + 3tu^2 - u^3$.

7. $y^4 - 7y^3 + 14y^2 + y - 21$.

8. $16a^4 - 32a^3 + 24a^2 - 8a + 1$.

9. $x^3 + 2x^2 + 3x + 2$.

10. $a^{3x} + 3a^{2x}b^y + 3a^xb^{2y} + b^{3y}$.

11. $x^a + y^a$, a being odd. 12. $8x^3 - 12x^2y + 6x^2y^2 - y^3$.

By using the factor theorem show that each of the following equations has the root given.

13. $x^3 + 12x^2 + 31x - 20 = 0$; $x = -5$.

14. $\dfrac{x^3 - 1}{x - 1} = \dfrac{b^3 - 1}{b - 1}$; $x = b$.

15. $y^3 + 2y^2 + 3y + 2 = 0$; $y = -1$.

16. $x^4 - 7x^3 + 14x^2 + x - 21 = 0$; $x = -1$.

Find a root to each of the following equations, using the Factor Theorem:

17. $x^3 + x^2 + x + 1 = 0$. 18. $x^4 - 4x^3 + 2x^2 + x + 6 = 0$.

19. $x^3 + 3x^2 - 5x - 10 = 0$. 20. $x^4 - 2x^2 - 3x - 2 = 0$.

21. $x^4 + 3x^3 + 4x^2 - 12x - 32 = 0$.

22. $3x^3 - 16x^2 + 23x - 6 = 0$.

23. $x^3 + x^2 + x + 1 = 0$.

24. Show that $(1 - x)^2$ is a factor of $1 - x - x^n + x^{n+1}$, n being a positive integer.

25. Show that $(x - 1)$ is a factor of $nx^{n+1} - (n + 1)x^n + 1$, n being any positive integer.

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GEBRA

80. **10.** 80.

; 36 in.
60°; $C=90°$.

00.
; $AC=8$ in.

8 in.

82.304.

4.

(*b*), 19.

ANSWERS

ANSWERS

TO

KENT'S A FIRST COURSE IN ALGEBRA

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§ 5, pages 2, 3

4. 10. **5.** 2. **6.** 21. **7.** 12. **8.** $\frac{3}{2}$. **9.** 380. **10.** 80.
11. 11. **12.** $\frac{7}{4}$. **13.** $\frac{61}{1200}$. **14.** 81. **17.** 120. **18.** 5.

§ 9, pages 7 to 9

1. \$200; \$100. **2.** 63; 7. **3.** 5.
4. *A*, \$10,000; *B*, \$5000; *C*, \$15,000.
5. 16 men; 8 women; 48 children. **6.** 12 in.; 24 in.; 36 in.
7. \$48; \$16; \$16. **8.** 420; 140; 70. **9.** $A=30°$; $B=60°$; $C=90°$.
10. 375 bu. of corn; 1125 bu. of wheat; 1500 bu. of oats.
11. 54, 782, 608.
12. N. Y., 4,500,000; Chicago, 2,250,000; St. Louis, 900,000.
13. Length, 80 rds.; width, 20 rds. **14.** $BC=AB=16$ in.; $AC=8$ in.
15. 40 feet. **16.** \$300.
17. Length, 300 feet; width, 160 feet. **18.** 3200. **19.** 8 in.
20. *A*, 80 miles; *B*, 20 miles.

§ 16, pages 10 to 12

1. 24, if $x=2$. **2.** 540; 360. **3.** 31. **4.** 1.
5. 7. **9.** 10. **10.** 400. **11.** 1382.304.
12. 3769.92. **13.** 37. **14.** $\frac{13122}{125}$. **15.** 144.
16. 1005.312 cu.ft. **17.** (*a*), 0; (*b*), 0; (*c*) 2. **18.** (*a*), 0; (*b*), 19.

§ 23, pages 17, 18

5. (*a*), 12; (*b*), 25. **6.** (*a*), 18; (*b*), 12.
7. (*a*), 8 to right; (*b*), 5 to right. **8.** (*a*), 6 below; (*b*), 6 above.
9. +8; +5. **10.** −6; +6.

§ 26, pages 19, 20

5. (*a*), −5°; (*b*), +5°. **6.** (*a*), −5°; (*b*), +5°.
7. (*a*), +127°; (*b*), +27°. **8.** (*a*), 35°; (*b*), −5°.
9. 8 B.C. **10.** $1000.

§ 27, pages 21, 22

3. (*a*), 10°; (*b*), −10°. **4.** (*a*), −2°; (*b*), +2°.
5. (*a*), 15°; (*b*), −10°. **6.** (*a*), 27°; (*b*), −15°.
7. $1000. **8.** $−1000. **9.** $1000. **10.** −$1000.
11. $-48\frac{1}{2}$. **12.** 74°. **13.** 74°; −74°. **14.** −21°.
15. 66°. **16.** $+3\frac{1}{3}$ feet. **17.** 8 miles up; 12 miles down.
18. 20+4; 20−4. **19.** (*a*), 100 miles per hour; (*b*), 20 miles per hour.

§ 32, pages 25 to 27

21. 5. **22.** −5000. **23.** −59. **24.** 10.
25. $-\frac{3}{5}$. **30.** +3°; −2°; −14°. **31.** −3°. **32.** 2000 years.
33. −49 or 49 B.C. **34.** 1209.
35. 5° south of the place; −345.8 miles north same place.
36. 19 yds. behind north goal. **37.** 21° higher at 7 P.M.
40. $35. **42.** −9.

§ 38, page 31

1. $-x$. **2.** a^2. **3.** $6a^2b$. **4.** $7xy$. **5.** $-\frac{1}{8}abc$.
6. $5(x-y)$. **7.** $-3.9\,a^2b^2c^2$. **8.** $\frac{1}{4}gt^2$. **9.** $1\frac{1}{4}\,s$. **10.** $11(a+b)$.
11. $-6(H+R)$. **12.** $-2(P-q)$. **13.** 3 times angle A.
14. 15 times 45°. **15.** $y(7a+6c-3b)$.
16. 11(−2°). **17.** $(a+b-c)\cdot$\$5. **18.** $(a-b-c)(x-y)$.
19. $(p+q+r)(-m)$. **20.** $(a'+a''+a''')gt^2$.

§ 39, page 32

1. $12x^3$. **2** $-4a^2b$. **3.** $-6t^2$. **4.** $-\frac{3}{2}xy$. **5.** $5mn$.
6. $-2ab$. **7.** $-(a+b)$. **8.** $2(x-y)$. **9.** $m+n$. **10.** $-2(x+y)$.
11. $\frac{a+b}{4}$. **12.** $-3.5cd$. **13.** $x-7$. **14.** $-8a^2bc$; $+8a^2bc$.
15. 5 times angle B. **16.** $\frac{9}{16}gt^2$. **17.** 7 times 30°.
18. $(a'-a'')gt^2$. **19.** $(a-b)$ \$5. **20.** 10°.

§ 44, pages 35 to 37

1. $11a^2b+10ac^2$. **2.** $25t^2-4t+12t^3$.
3. $-4x^4+6x^3-15x^2+6x+4$. **4.** $\frac{7}{4}gt^2+\frac{3}{4}at^2$.
5. $5a^2b-6xy-2cd$. **6.** $-7abc+9b^2c^2+15acd$.
7. $3a^4-2a^3b+10a^2b^2+7ab^3+b^4$. **8.** $-(a^2+b^2)+3(a^2-b^2)+c^2$.
9. $a^2c^2-\frac{5}{6}a^2b^2-\frac{1}{2}b^2c^2$. **10.** $-4xy$. **11.** $-a^2-6ab-b^2$.
12. $-3(x-1)-6(x+2)-3(x-4)+3(x-2)$.
13. $-x^4-3x^3y+3x^2y^2-3xy^3-y^4$. **14.** $9a-3b+3c-2d$.
15. $4x^2-4xy+y^2$. **16.** $4(a-x)-6(a+y)$.
17. $12a^2b+3c^2x-7a^2y$. **18.** $11(u+v)-(u-v)$.
19. $-x^2-2xy-y^2$. **20.** $x^2+2xy+y^2$. **21.** $-2n-p-q$.
22. $b+c+d+e$. **23.** $-x-y+z+4$. **24.** 35°.
25. $(a+b-c)$ degrees. **26.** $[x+y+z-(a+b)]$ dollars.
27. $(a+c-b-d)$ miles. **28.** $(20+a+b-c)$ degrees.
29. (*a*), $(1000-b)$ feet; (*b*), $(1000-b+c)$ feet.
30. $[2000+a+c-(1000+b+e)]$ feet.

§ 45, page 38

1. $2a-2b-2c$. **2.** $4a-20b$. **3.** $10mn-6pq$. **4.** $-5a^3+b^3$.
5. $5\frac{1}{4}a^2b^2$. **6.** $20-a+b-c-3d$. **7.** $b+5c+1$.
8. $13+11a-2b+7c$. **9.** $44a-50K+42L$. **10.** $-10a+11b+12c$.

§ 46, pages 39, 40

4. $-2y^2-7y+3$. **5.** n^2-n+2. **6.** $3(a-b)$. **7.** $8(r-s)^2$
8. $-10a-b$. **9.** $-5a+3b+c$. **10.** $2m+7n$.
11. $16x^2+xy-7y^2$. **12.** $2ax-4by+2cz$. **13.** $10x^2y^2-9xy$.
14. $4a-8y$. **15.** $2a^2-13a-223$. **16.** $(a+c-e)x+(b+d)y$.
17. $(2a+3b-4)x$. **18.** $(m-n-r)y+(p-q+t)x$.

19. $(5-a+3b-c)(x^2-y^2)$. **20.** $(3-4c)(a-x)$.
21. $(5-8cd-4)y-(3ac+4ab+5a-2c)x$.
22. $(a+g)x^3+(b+e-k)x^2-(e+d)x$.
23. $-(a+2b+1)x^3-(1+c)x^2-(1-b)x$. **24.** $2b$; $b+c$.
25. k and $2k-1$; or k and 1. **26.** k and $k+1$.
27. (*a*), $(a-b)+(c-d)$; (*b*), $(a^2+2ab+b^2)-(c^2+2cd+d^2)$.
28. (*a*), $(a-b)+(c-d)+(e-f)+(g-h)$;
(*b*), $-(b-a)-(d-c)-(f-e)-(h-g)$.

§ 54, pages 46, 47

1. $y=3$. **2.** $n=4$. **3.** $t=\frac{5}{2}$ **4.** $x=\frac{13}{14}$. **5.** $x=2$.
6. $y=11\frac{1}{6}$. **7.** $r=\frac{5}{2}$. **8.** $v=32$. **9.** $m=-15$. **10.** $x=5$.
11. $x=8$. **12.** $x=-\frac{3}{14}$. **13.** $x=7$. **14.** $y=4\frac{22}{35}$. **15.** $y=-\frac{31}{8}$.
16. $k=-\frac{28}{31}$. **17.** $e=\frac{7}{54}$. **18.** $k=+1$. **19.** $t=\frac{3}{80}$. **20.** $x=2$.

§ 57, pages 49 to 54

1. 155; 91. **2.** $\frac{s+d}{2}$; $\frac{s-d}{2}$. **3.** (*a*), 30, 15; (*b*), −7, −3.
4. 3, 4, 5, 6. **5.** $\frac{96}{7}$, $\frac{48}{7}$, $\frac{24}{7}$. **6.** 5, 15, 10. **7.** 8.
8. 4, 5, 6, 7, 8. **9.** 10″, 11″, 12″, 13,″ 14″.
10. $\frac{18}{31}$, $1\frac{5}{31}$, $2\frac{10}{31}$, $4\frac{20}{31}$, $9\frac{9}{31}$ inches.
11. Panama, 49 miles; Suez, 100 miles.
12. Miss., 3160; Mo., 3100; Amazon, 3300; Nile, 3400 miles.
13. 1950. **14.** 400,002. **15.** Width 15 ft.; length 37 ft.
16. Width, single court 27 ft.; double court 36 ft.; length 78 ft.
17. 14. **18.** John $8\frac{1}{2}$ yrs. Henry 22 yrs.
19. $A=52\frac{1}{2}$; $B=26\frac{1}{4}$; $C=26\frac{1}{4}$. **20.** 5 miles; 1 mile; 12 hrs.
21. 2 mi. per hr. **22.** $8\frac{4}{7}$ miles. **23.** 180 miles.
24. 1200 miles. **25.** \$84; \$84; \$282. **26.** \$7833⅓; \$7333⅓; \$6833⅓.
27. 5250 miles. **28.** 143,578.
29. 1,057,143; 707,143; 707,143; 528,571.

30. N. Y., $36,000,000; Michigan, $18,000,000; Chi., $12,000,000.
31. 840,000 miles. **32.** $a=c=3.508$; $b=4.984$.
33. $A=45°$. **34.** 45°; 90°; 45°.
35. $a=6$; $b=8$; $c=10$. **36.** $A=36°$; $B=54°$; $C=90°$.
37. $A=B=C=60°$; $a=b=c=4$; (Fig. 13), $a=b=c=8$.
38. $A=36\frac{12}{13}°$; $B=81\frac{7}{13}°$; $C=61\frac{7}{13}°$. **39.** $a=8\frac{1}{3}$; $b=12\frac{1}{3}$; $c=10\frac{1}{3}$.
40. $A=40°$; $B=80°$; $C=60°$.

§ 66, pages 64 to 66

1. $x=6$; $y=2$. **2.** $x=4$; $y=3$. **3.** $x=6$; $y=3$.
4. $x=-6$; $y=5$. **5.** $x=-2\frac{12}{37}$; $y=2\frac{11}{37}$. **6.** $v=5$; $a=1.5$.
7. $s=10$; $v=1$. **8.** $u=10$; $t=2$. **9.** $m=1$; $n=1$. **10.** $v=6$; $t=9$.
11. $r=\frac{35}{13}$; $s=-\frac{72}{13}$. **12.** $s=25.75$; $t=12$. **13.** $n'=17$; $n''=7$.
14. $\frac{s+d}{2}$; $\frac{s-d}{2}$. **15.** 50 acres each. **16.** 60 children; 190 adults.
17. Width 40; length 80. **18.** 200 men. **19.** 145 and 179.
20. 45 at $1.80; 5 at $3. **21.** $4\frac{4}{5}$ minutes.
22. 40 lbs. at 30¢.; 10 lbs. at 40¢. **23.** $3\frac{3}{4}$ qts. **24.** 400 lbs.
25. Width 160; length 300. **26.** $128\frac{4}{7}$; $171\frac{3}{7}$ miles.
27. 3,000,000 sq. miles. **28.** $3\frac{1}{4}$ lbs.
29. 800 tons ore; 4800 tons limestone; 14,400 tons coke.
30. $1600; $2100.

§ 72, pages 72, 73

1. x^4-y^4. **2.** $6a^4b-2a^2b^3+2ab^4+3a^3b^3-ab^5+b^6$.
3. $1-3x+5x^2-3x^3$. **4.** $16a-16b+a^4b-b^5$.
5. $a^6-4a^5b^2+2a^4b-2a^3-4a^3b^3+4a^3b^2-a^4+a^2b^2-a^2b-2ab+2a$.
6. $m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6$.
7. $x^5+2x^4y+x^3y^2+x^2y^3-y^5$.
8. $v^4t^2+v^3t^3-3v^3t^2+4v^3+v^2t^4-3v^2t^3+4v^2t+vt^2-3vt^4+9t^2-12$.
9. $4x^6-8x^5y-3x^4y^2+24x^3y^3-3x^2y^4-18xy^5+9y^6$.
10. $-\frac{a^2t^6}{2}-\frac{3a^2t^5}{4}-\frac{a^2t^4}{2}-3a^2t^3+2a^2t^2$. **11.** $\frac{b^2}{4}+\frac{2bc}{6}-\frac{c^2}{9}+\frac{d^2}{16}$.
12. $a^{4n}-2a^{2n}b^{2n}+b^{4n}$. **13.** $3^{2x}-2^{2y}$. **14.** $5^{2c}-5^c\cdot a^e-2a^{2e}$.
15. $a^4+\frac{a^3}{2}-\frac{a^2}{2}+\frac{3a}{8}-\frac{1}{16}$. **16.** x^3+y^3.
17. $-cay^2+ady+c^2y^3-cdy^2$. **18.** $x^4+x^2y^2+y^4$. **19.** $a^{2x}-b^{2y}$.
20. $a^{3x}+a^xb^{2y}-a^xc^{2p}+a^{2x}b^y+b^{3y}-b^yc^{2p}+a^{2x}c^p+b^{2y}c^p-c^{3p}$.

§ 74, pages 75, 76

1. $5a^2b^2c$. **2.** $-4m^4n^2$. **3.** $8x^5y^3$. **4.** $-ab^3$. **5.** 0.
6. $9a^7b^3$. **7.** 9. **8.** $-(x-y)^4$. **9.** $(a^2-b^2)^2$.
10. a^m. **11.** a^{m-c}. **12.** $-5x^{a-b}y^{-c}$. **13.** $-3a^{m-1}c^{n-1}$.
14. $3x^{2a-2}y^{c-1}$. **15.** $-m^2(x^2-y^2)^2$. **16.** $5a^5b^6$.
17. $-4m^4n^3$. **18.** $3a^{3m-2}c^{4n-1}$.
19. $a^{2+x-ax}b^2(x^m-y^m)^3$. **20.** $x^{3m}\cdot z^3$.

§ 75, page 77

1. $x^2-3xy+3y^2$. **2.** $bc-5ab^2+4b^2c^2$.
3. $4(b^3-a^3)^3+3(x^2+y^2)+6(m^4-n^4)$. **4.** $(x^m-y^n)^2+1-(x^m-y^n)$.
5. $-a^{m-1}b^{n-1}c^{n-1}+2a^{2m-1}b^{3n-1}c^{4n-1}-4$.
6. $\frac{3}{2}mn+2c+4m^2n^2c^3$. **7.** $v+at-3at^2$. **8.** $n+bx^{n-c}+dx^{m-c}$.
9. $a^{-n-1}+5a^{1-m}-3a^{n-m-1}b-a^{m+n-1}$.
10. $-10xy^{m-3}-x^{a-2}y^{-1}+7x^{c-2}y^{-3}-2x^{-2}y^{s-3}$.

§ 76, pages 80, 81

1. $m-3$. **2.** $a+3$. **3.** $8a^2b^2-1$. **4.** $8x-3y$. **5.** y^2-y-12.
6. $v-3a$. **7.** a^3-a^2+3a-5. **8.** Rem. $\frac{2(11x-10)}{27(3x^2-x+2)}$.
9. Rem. $\frac{198xy^3+1926y^4}{7(7x^2-4xy-9y^2)}$. **10.** $9m^2+6m+1$.
11. $a^{2m}-2a^mb^n+b^{2n}$. **12.** Rem. $\frac{m^a}{m^{a+1}+1}$. **13.** $a^{2m}+a^mb^n+b^{2n}$.
14. $\frac{a^2}{4}+\frac{ab}{3}+\frac{b^2}{9}$. **15.** $4a^4+6a^2v+9v^2$. **16.** m^2-mn+n^2.
17. Rem. $\frac{6y^2}{x+y}$. **18.** x^a+3. **19.** $a^{3p}-3$. **20.** $x^4+x^3y+x^2y^2+xy^3+y^4$.

§ 77, page 83

1. $x=7;\ y=1$. **2.** $v=\frac{3}{4};\ s=\frac{2}{3}$. **3.** $x=5;\ y=1$.
4. $t=5;\ u=10$. **5.** $x=-5\frac{1}{2};\ y=-8$. **6.** $m=1.5;\ n=-0.08$.
7. $x=11;\ y=5$. **8.** $r=4;\ s=-5$. **9.** $x_1=14;\ x_2=\frac{7}{6}$.
10. $m=\frac{1}{2};\ n=3$. **11.** $r_0=-10;\ r_1=1.4$. **12.** $x=-2;\ y=8$.

13. $s^1=4\frac{1}{7}$; $s=15\frac{4}{7}$. **14.** $x=5$; $y=3$. **15.** $y=\frac{1}{2}$; $z=-9$.

16. $n=1$; $s=\frac{1}{3}$. **17.** $x'=\frac{4}{5}$; $y'=\frac{6}{5}$. **18.** $x'=6$; $x''=8$.

§ 78, pages 83 to 91

1. 11. **2.** N. Y. C. \$$34\frac{1}{3}$; I. C. \$199.
3. $4\frac{4}{5}$ feet from end carrying the 18-lb. weight.
4. $w'=6$ lbs.; $w''=2$ lbs. **5.** $w'=2\frac{2}{5}$ lbs.; $w''=1\frac{3}{5}$ lbs.
6. $R=25$ mi. per hour; dist. $=60$ miles. **7.** 10 ft.; 12 ft.
8. $109\frac{1}{11}$ lbs. **9.** $52\frac{4}{23}$ lbs. **10.** $83\frac{1}{3}$ lbs. **11.** $41\frac{2}{3}$ lbs. **12.** $83\frac{1}{3}$ lbs.
13. $3\frac{3}{5}$ inches, if $OB=36$ inches. **14.** $29\frac{1}{3}$ ft.; $29\frac{1}{3}$ ft.; $26\frac{1}{3}$ ft.
15. $55{,}666\frac{2}{3}$ sq.ft. upper lock; $53{,}666\frac{2}{3}$ sq.ft. other locks.
16. 20 at Gatun; 46 in all. **17.** \$5,374,500.
18. Length 20 ft.; width 18 ft. **19.** Length 15 ft.; width 2 ft.
20. $1178\frac{4}{7}$ ft. per second. **21.** 15 miles. **22.** 9:$22\frac{1}{2}$ A.M.
23. $t=.9596$ of a second. $l=3.25$ ft.
27. 30.33 days. **28.** 12 ft.
29. 3 ft., 9 in. **30.** 12.
31. Horse 7; donkey 5. **32.** Length 300 ft.; width 160 ft.
33. Width 7 ft.; length 20 ft. **34.** 20 ft.; 10 ft. **35.** 15ft.; 9ft.
36. 70 children. **37.** 12 rds.; 6 rds.
38. 550 ft.; 1100 ft. **39.** 1000 ft.; 650 ft. **40.** 14, 16, 26.
41. Rate $46\frac{2}{3}$ miles per hour; distance, $466\frac{2}{3}$ miles.
42. $1\frac{7}{8}$, A's rate; 3, B's rate. **43.** $8\frac{1}{3}$ rds.; 19 rds.
44. Length 200 rds.; width 160 rds. **45.** $a=30$; $b=20$; $c=30$.
46. $a=30$; $b=20$: $c=30$. **47.** 10; 5. **48.** $a=28$; $b=20$; $c=12$.
49. 48, 28, 5. **50.** $A=30°$; $B=90°$; $C=60°$.

§ 103, pages 125 to 127

1. 9 and 4. **2.** 12 ft. and 8 ft. **3.** 25 ft. **4.** $4\frac{2}{5}$; $6\frac{2}{5}$; $1\frac{7}{15}$.
5. $41\frac{1017}{3375}$ cu.ins. **6.** 25 miles per hour. **7.** 20. **8.** 36 in. **9.** 5 in.
10. Base 10 in.; alt. 12 in. **11.** Base 12 ft.; alt. 8 ft. **12.** 15 ft.

§ 107, page 136

1. 31. **2.** 49. **3.** 24. **4.** 0.474+. **5.** 0.21. **6.** 0.01
7. 3.952+. **8.** 0.0316+. **9.** 0.790+. **10.** 1.7320+. **11.** 0.816+.
12. −3.128.− **13.** −4. **14.** −2.483+. **15.** 0.408+.

§ 110, pages 139, 140

1. $\sqrt[5]{x^3}$. **2.** $\sqrt[3]{x^2}$. **3.** $\sqrt[4]{x^3}$. **4.** $\sqrt[3]{y^8}$. **9.** $\sqrt[3]{4(x^a-y^b)^2}$.
10. $\sqrt{729ab^5}$. **11.** 8. **12.** 243. **13.** 9. **18.** 0.0625. **19.** 4096.
20. 432. **21.** $2(81)^{\frac{1}{3}}$. **22.** $(56x)^{\frac{5}{3}}$. **31.** $a^{\frac{n}{5}}b^{\frac{p}{5}}$. **32.** $a^{\frac{5}{n}}b^{\frac{6}{n}}$.
35. a^2b. **39.** $(a+b)^{\frac{m}{n}}$. **40.** $(a+b)^{\frac{p}{m}}(a-b)^{\frac{q}{n}}$.

§ 113, pages 143, 144

1. $2x\sqrt[3]{2x}$. **2.** $\pm 2m\sqrt{2m}$. **7.** $-5t\sqrt[3]{t}$. **10.** $\pm 5\sqrt{\sqrt[3]{4}}$ or $5\sqrt{(4)^{\frac{1}{3}}}$.
17. $2(a+b)(a-b)^2\sqrt{2}$. **18.** $\pm(x^2-y^2)\sqrt{x^2-xy+y^2}$.
19. $\pm\frac{R}{2}\sqrt{2}$. **20.** $\pm 5\sqrt{\sqrt{5}+4}$. **21.** $\sqrt{75}$. **22.** $\sqrt{18}$.
30. $\sqrt{a-b}$. **31.** $\sqrt{5}$. **34.** $\sqrt{5}$. **43.** $\frac{1}{3}\sqrt{6}$. **44.** $\frac{1}{3}\sqrt{6}$.
45. $\frac{1}{3}\sqrt[3]{18}$. **46.** $2\sqrt{xy(x-y)}$. **47.** $\frac{(a+b)^2}{(a-b)^2}\sqrt{a^2-b^2}$.
48. $(a^2-b^2)\sqrt{a^2-b^2}$. **49.** $\frac{1}{2}\sqrt[5]{10}$. **50.** $\frac{1}{a+b}\sqrt{a^2-b^2}$.

§ 115, page 145

1. $32\sqrt{3}$. **2.** $7\sqrt[3]{2}$. **3.** $6\sqrt[3]{4}$. **4.** $7\sqrt{5}$.

§ 122, pages 153 to 155

26. 18. **27.** $\frac{1}{2}$. **28.** 2.5. **29.** $\frac{1}{4}$. **30.** 8.
31. $\frac{1}{4}$. **32.** 1. **33.** 1. **34.** 4.5.
35. $\frac{ab(b^3+a^3)}{b^4-a^4}$. **36.** $\frac{bc}{a^2(b+c)}$. **37.** $\frac{1}{x^3+y^3}$. **38.** $23\frac{81}{85}$. **39.** $\frac{1}{16r^2s^2t^2}$.
40. $243b$. **41.** 2. **42.** a^3. **43.** 5. **44.** $\frac{1}{2}$.
45. $\frac{x^3+1}{x^2(x+1)}$. **46.** 4.33. **47.** .707+. **48.** 108.35.
49. 4.98. **50.** $A=\frac{h^2}{\sqrt{3}}$. **51.** .816. **52.** 1.414+.
53. .894. **54.** .576+.

§ 126, pages 160 to 164

1. 8.484 ft. **2.** 117.15 **3.** B, 10 miles per hour.
4. 65.2 hrs. **5.** 14.58 sec. **6.** 42.6. **7.** 3 in.
8. Length 60; width 30. **9.** 10 ft. **10.** 20 rds. **11.** 3.515 inches.
12. 3.53+ feet. **13.** 567. **14.** $p=\frac{62\frac{1}{2}d^2l}{2}$.
15. 11.69 ft.; 29.225 ft. **16.** 10¢ per lb. **17.** 10 ins.; 20 ins.
18. 5 hrs. **19.** 4.33+ hrs. after B starts.
20. Length 8 inches; width 6 inches. **21.** Length 10; width 8.
22. 8. **23.** +3 or −5. **24.** 2, 3, 4, 5.
25. A, 100; B, 66⅔. **26.** 3.51 ft. **27.** 211.6+ ft. per sec.
28. Length, 9.62 in.; width, 4.81 in.
29. Base 12; area 48. **30.** $h=13.856$.

§ 128, page 168

1. Real, unequal. **2.** Real, equal. **3.** Real, equal.
4. Real, equal. **5.** Imaginary. **6.** Real, unequal.
7. Real, unequal. **8.** Real, unequal. **9.** Real, unequal.
10. Imaginary.

§ 133, pages 176, 177

1. $y=12$. **2.** $s=36$. **3.** $x=44\frac{1}{12}$. **4.** $v=\frac{3a}{2}$. **5.** $x=6$.
6. $x=5$. **7.** $x=13$. **8.** $x=9$. **9.** $x=-1$ or -6. **10.** $y=6$.
11. $x=\frac{b^4+2b^3+b^2(1+2a)-2ab+a^2}{4b^2}$. **12.** $y=149+42\sqrt{11}$.
13. $x=2$. **14.** $x=3$ or -11. **15.** $x=13$.

§ 141, page 186

1. $a+b+\frac{b^2}{a}$. **2.** $a-b-\frac{b^2}{a+b}$. **3.** $x-y-\frac{2y^2}{x-y}$.
4. x^2+xy-y^2. **5.** a^2-ab+b^2. **6.** $r+\frac{p^2}{r-p}$.
7. $a^2+2ab+b^2+\frac{b^3}{a+b}$. **8.** $9a^2+12ab^2+16b^4$.
9. $x^2-xy+y^2-\frac{2y^3}{x+y}$. **10.** x^2-xy+y^2.

§ 142, pages 188, 189

9. $\frac{2(m^2+1)}{m^2-1}$. **10.** $\frac{5(5x-4y)}{12}$. **11.** $\frac{3x^2+1}{(x^2-1)(x-1)}$.
12. $\frac{5m^2-2mn+n^2-8m+8n}{4(m-n)^2}$. **13.** $\frac{1-2y}{y^2+y}$. **14.** $\frac{m^2+m-6}{6(m^2-1)}$.
15. $\frac{28t+1}{2(4t^2-1)}$. **16.** $\frac{4a-1}{a^2-1}$. **17.** $\frac{x^3+x^2y-xy}{(x+y)^3}$.
18. $\frac{a(a+13)}{(a+1)(4a^2-1)}$. **19.** $\frac{a^2-13+4a}{2(a^2-1)}$. **20.** $\frac{2a^2-9a+44}{a^3+64}$.

§ 145, pages, 190, 191

1. $\frac{n^2}{6m}$. **2.** $9ax$. **3.** $\frac{10xy^2c}{b}$. **4.** $\frac{(y-1)(y-6)}{y^2}$. **5.** $\frac{m-5}{m-7}$.
6. $\frac{2}{a-b}$. **7.** $x(x+y)$. **8.** $\frac{1}{y}$. **9.** $\frac{a-4}{a+4}$.
10. $\frac{3(a-b)}{2b(a+b)}$. **11.** $\frac{b^2-8}{(b-2)(8-b^3)}$. **12.** 1. **13.** 1.
15. $\frac{mn(m-4n)}{(m+n)(m^2+n^2)}$. **16.** $-\frac{25000a^3x}{243c^9}$. **17.** $\frac{a^5b^4}{(a^4-a^2b^2+b^4)^2}$.
18. $\frac{405}{128a^5}$. **19.** $\frac{b(b-y)^2}{(p-q)(b^3+b^2y+by^2+y^3)}$. **20.** $\frac{y-x}{y+x}$.

§ 154, pages 200 to 202

1. $2\frac{2}{3}$ and $3\frac{1}{3}$. **2.** $60\frac{12}{43}$ ft.
3. 18 in.; 27 in.; 36 in. **4.** 8; 12.
5. $12 and $18. **6.** $6\frac{4}{9}$ ft. from end which holds the 60-lb. boy.
7. $18\frac{3}{4}$ in. from end supporting the 6-lb. weight.
8. 16.914; 20.715; 25.371. **9.** 12; 14. **10.** 24; 36; 48.
11. 20. **12.** $\frac{1}{3}$. **13.** 300 ft.

§ 155, pages 205, 206

23. $x(x^2-xy+y^2)$; $y(x^2-xy+y^2)$ gals. respectively. **24.** $80; $120.
25. 20 ft.; 30 ft. **26.** 11.18 inches. **27.** $s=\frac{1}{2}d\sqrt{2}$; $d=s\sqrt{2}$.
28. $s=10\sqrt{2}=14.14+$. **29.** $d=12\sqrt{2}$; or 16.968+.
30. $\frac{a}{2}\sqrt{3}$; area $=\frac{a^2}{4}\sqrt{3}$. **31.** $h=4$; $s=\frac{8}{3}\sqrt{3}$.
32. $h=a\sqrt{3}$; $s=2a$. **33.** $p-q$. **34.** $\frac{k+n}{k-n}$.

35. $a^4+a^3b+a^2b^2+ab^3+b^4$. **36.** $\frac{x^2-y^2}{xy^2}$. **37.** $\frac{1}{2a^2-1}$.

38. $\frac{a}{a-1}$. **39.** $\frac{1}{6}\sqrt{6}$. **40.** $\frac{2}{5}\sqrt{5}+\sqrt{3}$.

§ 159, pages 209 to 211

1. $x=15$. **2.** $x=-3$. **3.** $h=2$. **4.** $n=5$. **5.** $y=1$.
6. $e=12$. **7.** $k=13$. **8.** $x=0$. **9.** $x=5$. **10.** $y=-6\frac{3}{7}$.
11. $x=65$. **12.** $y=-1$. **14.** $r=\frac{25}{11}$; $s=\frac{25}{2}$. **15.** $u=\frac{3}{2}$; $v=\frac{2}{3}$.
16. $x=\frac{1}{2}$; $y=-1$. **17.** $x=-23\frac{7}{11}$; $y=-10\frac{3}{11}$.
18. $x=6$. **19.** $v=\frac{9}{26}$. **20.** $y=3$; $z=-5$.

§ 162, pages 213 to 215

1. $x=2a$. **2.** $y=\frac{b(a-1)}{a-b}$. **3.** $y=abc$.

4. $x=\frac{a+b}{2}$; $y=\frac{a-b}{2}$. **5.** $n=\frac{s+d}{2}$; $n'=\frac{s-d}{2}$. **6.** $e=\frac{a+c}{3}$; $k=\frac{a-2c}{3}$.

7. $x=\frac{m+n}{f+e}$; $y=\frac{n(m+n)}{nf-me}$. **8.** $x=a+b$.

9. $x=\frac{e'c-ec'}{e'k-ek'}$; $x'=\frac{ck'-c'k}{e'k-ek'}$. **10.** $y=mnp$.

11. $x=\frac{2(d-b)}{ad-bc}$; $y=\frac{2(c-a)}{bc-ad}$. **12.** $x=\frac{5c+2a}{ac}$; $y=\frac{5c-2a}{ac}$.

13. $x=\frac{ab}{a+1}$; $y=\frac{b}{a+1}$. **14.** $r=\frac{a+2ab-b}{a^2+b^2-a^2b}$. **15.** $y=\frac{1}{a-b}$.

16. $t=-\frac{v}{2g}\pm\frac{\sqrt{v^2+4gs}}{2g}$. **17.** No solution. Why?

18. $x=\frac{2n}{n^2+1}$; $y=\frac{2n}{1-n^2}$. **19.** $x=\frac{35b}{6}$; $y=35a$.

20. $v=\frac{5ab}{2(a-b)}$; $v'=\frac{5ab}{4a+b}$. **21.** $t=1\pm\sqrt{2(m+1)}$.

22. $s=a$. **23.** $y=-1$ or $\frac{-2a}{a+b}$.

24. $n=\pm\frac{\sqrt{8s+(2a+d-1)^2}-2a-d+1}{2}$. **25.** $x=\frac{ab-1}{be+d}$.

26. (*a*) $r=\frac{sl-a}{s}$; (*b*) $r=\frac{sl-a}{s-l}$. **27.** (*a*) $s=\frac{4}{9}$; (*b*) $s=-\frac{1}{9}$.

28. $r=\frac{e-CR}{Cn}$. **29.** $n=\frac{CR}{e-Cr}$. **30.** $a=\frac{1}{2-n}$.

31. $P''=\frac{P'V''}{V'}$. **32.** $R=\frac{r''r'r}{r''r'+r''r+r'r}$.
33. $x=\frac{d+ef}{ae+1}$; $y=\frac{ad-f}{ae+1}$. **34.** $x=\frac{bk+cd}{5c+4k}$; $y=\frac{cd+bk}{5b-4d}$.

§ 163, pages 216, 217

1. $\frac{s+d}{2}$; $\frac{s-d}{2}$. **2.** $\frac{a+d}{2}$; $\frac{a-d}{2}$. **3.** $\frac{mc}{m+n}$ in. and $\frac{nc}{m+n}$ in.
4. $\frac{d}{a+b}$. **5.** $\frac{an(a+b)}{bn-a^2}$.
6. 10.76+ miles from Oklahoma City. **7.** $9\frac{2}{13}$ miles. **8.** $5\frac{1}{3}$ days.
9. 48. **10.** 44 and 45. **11.** 26 yrs. **12.** 90 miles. **13.** 3 and 4.
14. 4125+ sq. in. **15.** $782\frac{2}{3}$ ft. **16.** 8492.

§ 164, pages 218 to 220

1. $d_1=\frac{d_2I_2}{I_1}\sqrt{\frac{C_1}{C_2}}$. **2.** 3.23 miles. **3.** 34.14 ft. **4.** 6.83+ft.
5. 5.06+ ft. **6.** 606,060.6 H.P. **7.** 5 ft. 5 in. **8.** 2043.8+ ft.
11. 480 miles. **12.** 90 miles. **13.** $1\frac{2}{3}$ and $3\frac{1}{3}$ miles per hour.
14. 120 ft. **15.** 75 words per minute. **16.** $x=16$.
17. 8″. **18.** 189.57 ft. **19.** 6.275 miles; 16.275 miles; 17.45 miles.

§ 167, pages 225, 226

1. 25; 35; 65. **2.** $a=25$; $b=23$; $c=18$. **3.** $17\frac{1}{2}$; $17\frac{1}{2}$; 25.
4. $33\frac{3}{4}°$; $33\frac{3}{4}°$; $112\frac{1}{2}°$. **5.** $A=79°$; $B=62°$; $C=39°$. **6.** 5; 15; 25.
7. $x=\frac{a+b-c}{2}$; $y=\frac{a-b+c}{2}$; $z=\frac{b+c-a}{2}$. **8.** 40, 20, and 10.
9. $A=73\frac{1}{3}°$; $B=63\frac{1}{3}°$; $C=43\frac{1}{3}°$. **10.** $d=12$ in.; $b=16$ in.; $l=20$ in.

§ 170, pages 229, 230

1. $l=8$ ft; $w=6$ ft.; $d=2$ ft. **2.** $l=30$ ft.; $w=10$ ft.
3. $l=100$ ft.; $w=50$ ft. **4.** 2 and 3. **5.** 6 and 2.

§ 171, page 231

1. $x=\pm 5$; $y=\pm 4$. **2.** $s=\pm 1$; $t=\pm 2$.
3. $n=\pm 2$; $m=\pm 3$. **4.** 5 and 13. **5.** 6 and 8.

§ 176, page 237

1. 16 and 12. 3. 3 and 5. 4. 15 and 11. 5. 6 and 8.
6. 9 and 4. 8. 3.97. 9. 10 and 6.
10. (*a*) 3 from end of leg 3; 2 from end of leg 4.
(*b*) Mid-point of hypotenuse.

§ 177, pages 238, 239

1. 35 ft. 2. 15, 20, and 25. 3. 4 and 5.
4. $\frac{3}{4}$ or $\frac{4}{3}$. 5. $l=20$ yds.; $w=1\frac{3}{5}$ yds. 6. 1, 2, and 3.
7. 4 rds. 8. 10 rds. 9. 12.649 ft.; 6.3245 ft.
10. 15 ft.; 8 ft.

§ 178, pages 239, 240

1. 787.92 ft. 2. 2598 ft. 3. 2887.92 ft.
4. 0.522″; also 11.9″. 5. 402 ft. 6. 188 ft. per sec.
7. 5.955 times farther on earth.
8. 75,248 ft. earth; 448,148+ ft. moon.
9. 2315.52 ft. 10. 4.2 ft.

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